Volume IIIC

Explanations,
Finite Element FORMA
Subroutines

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Expansion and Improvement of the FORMA System for Response and Load Analysis

MARTIN MARIETTA

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EXPANSION AND IMPROVEMENT OF THE FORMA SYSTEM FOR RESPONSE AND LOAD ANALYSIS

Volume IIIC - Explanations, Finite Element FORMA Subroutines

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Author:

Richard L. Wohlen

Approved:

Richard L. Wohlen

Richard L. Wohlen
Program Manager

Prepared for: National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

MARTIN MARIETTA CORPORATION
Denver Division
Denver, Colorado 80201

FOREWORD

This report presents results of the expansion and improvement of the FORMA system for response and load analysis. The acronym FORMA stands for FORTRAN Matrix Analysis. The study, performed from 16 May 1975 through 17 May 1976 was conducted by the Analytical Mechanics Department, Martin Marietta Corporation, Denver Division, under the contract NAS8-31376. The program was administered by the National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Huntsville, Alabama under the direction of Dr. John R. Admire, Structural Dynamics Division, Systems Dynamics Laboratory.

This report is published in seven volumes:

Volume I - Programming Manual,
Volume IIA - Listings, Dense FORMA Subroutines,
Volume IIB - Listings, Sparse FORMA Subroutines,
Volume IIC - Listings, Finite Element FORMA Subroutines,
Volume IIIA - Explanations, Dense FORMA Subroutines,
Volume IIIB - Explanations, Sparse FORMA Subroutines, and
Volume IIIC - Explanations, Finite Element FORMA Subroutines.

CONTENTS

ABSTRACT

This report presents techniques for the solution of structural dynamic systems on an electronic digital computer using FORMA (FORTRAN Matrix Analysis).

FORMA is a library of subroutines coded in FORTRAN IV for the efficient solution of structural dynamics problems. These subroutines are in the form of building blocks that can be put together to solve a large variety of structural dynamics problems. The obvious advantage of the building block approach is that programming and checkout time are limited to that required for putting the blocks together in the proper order.

The FORMA method has advantageous features such as:

1. subroutines in the library have been used extensively for many years and as a result are well checked out and debugged;
2. method will work on any computer with a FORTRAN IV compiler;
3. incorporation of new subroutines is no problem;
4. basic FORTRAN statements may be used to give extreme flexibility in writing a program.

Two programming techniques are used in FORMA: dense and sparse.

ACKNOWLEDGMENTS

The editor expresses his appreciation to those individuals whose assistance was necessary for the successful completion of this report. Dr. John R. Admire was instrumental in the definition of the program scope and contributed many valuable suggestions. Messrs. Carl Bodley, Wilcomb Benfield, Darrell Devers, Richard Hruda, Roger Philippus. and Herbert Wilkening, all of the Analytical Mechanics Department, Denver Division of Martin Marietta Corporation, have contributed ideas, as well as subroutines, in the formulation of the FORMA library.

The editor also expresses his appreciation to those persons who developed FORTRAN, particularly the subroutine concept of that programming tool.

LIST OF SYMBOLS

$[\ ]$	matrix	
$\{\ \}$	column matrix	vector
$\{\ \}^T$	row matrix	
T	transpose (when symbol is a superscript)	
$[\ ]_{m \times n}$	m designates the row size of matrix n designates the column size of matrix	
a_{ij}	a designates an element of matrix $[A]$ i designates the ith row of matrix $[A]$ j designates the jth column of matrix $[A]$	

I. INTRODUCTION

This volume presents an explanation of the function of each finite element subroutine in the FORMA library. Example problems are given in some cases to clarify the operations performed by a subroutine.

II. SUBROUTINE EXPLANATIONS

The subroutines are given in alphabetical order with numbers coming before letters.

Subroutine AXIAL calculates (on option) finite element: (1) mass matrices; (2) stiffness matrices (same as global load transformation matrices); (3) local load transformation matrices; (4) stress transformation matrices; and (5) vectors to locate the DOF (degrees of freedom) of the above matrices in the global DOF; for axial rod elements. The above matrices and vectors are written on disk units and constitute the output from this subroutine. All matrices are in dense programming logic.

Each mass and stiffness matrix, size 6x6, is in the global coorinate directions. The global coordinate order for each element is (U,V,W) joint 1, then joint 2 where U, V, W are translations. If the Euler angles are zero at a joint, then $U = \delta_X$, $V = \delta_Y$, $W = \delta_Z$.

Each global load transformation matrix, size 6x6, relates loads at the rod ends in the global coordinate directions to deflections in the global coordinate directions. The row order in this matrix is (P_U, P_V, P_W) joint 1, then joint 2 where P is force.

Each local load transformation matrix, size 2x6, relates loads at the rod ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is P_{x1}, P_{x2} where P_x is axial force.

Each stress stransformation matrix, size 2x6, relates stresses at the rod ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is σ_{x1}, σ_{x2} where σ is normal stress.

Each location vector (IVEC) locates the DOF of each finite element in the global DOF. For example, IVEC(6)=834 places element DOF 6 into global DOF 834. IVEC(3)=0 omits element DOF 3 from global DOF. This constrains element DOF 3 to zero motion.

The above matrices are calculated by using joint data and element data. The joint data is obtained from three matrices input to this subroutine which are (1) joint global X, Y, Z locations; (2)joint global DOF numbers; and (3) joint Euler angles.

The element data read in this subroutine is (1) options for mass, stiffness, local load transformations, stress transformations; (2) element material properties; and (3) element joint numbers, cross-sectional area.

Each mass matrix is calculated by transfer to subroutine MAS1A. Each stiffness matrix, loads, and stress transformation matrix is calculated by transfer to subroutine STF1A.

Subroutine B1A1 calculates a buckling (sometimes referred to as geometrical stiffness, initial stress, or stability) matrix for an axial rod element with unrestrained boundaries. The buckling matrix is based on a unit axial load. The buckling matrix is in the local coordinate system of the rod.

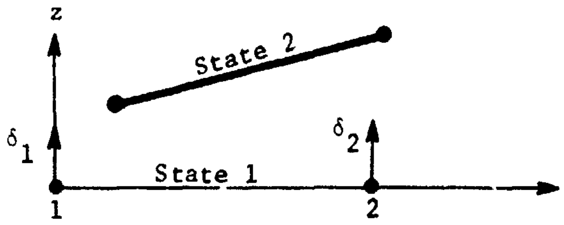
DESCRIPTION OF TECHNIQUE

From *Theory of Matrix Structural Analysis* by J. S. Przemieniecki, McGraw-Hill 1968, we obtained the buckling matrix. The strain energy for buckling is obtained as

$$U = \frac{1}{2} \, [\delta_1 \; \delta_2] \, \frac{F}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \qquad [1]$$

where the kernel matrix is the buckling matrix. A unit axial load of F = 1 is assumed here. L is the rod length.

The degrees of freedom are shown in the following sketch.



Subroutine B1A2 calculates a buckling (sometimes referred to as geometrical stiffness, initial stress, or stability) matrix for a beam element with unrestrained boundaries. The buckling matrix is based on a unit axial load. The buckling matrix is in the local coordinate system of the beam.

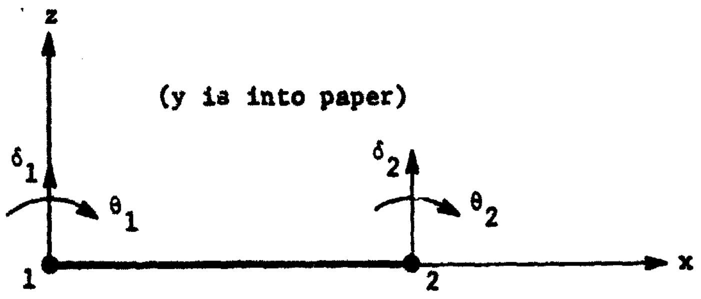
DESCRIPTION OF TECHNIQUE

From *Theory of Matrix Structural Analysis* by J. S. Przemieniecki, McGraw-Hill 1968, we obtained the buckling matrix. The strain energy for buckling is obtained as

$$U = \frac{1}{2} [\delta_1 \ \delta_2 \ \theta_1 \ \theta_2] \ F \begin{bmatrix} 6/5L & -6/5L & -1/10 & 1/10 \\ & 6/5L & 1/10 & 1/10 \\ & & 2L/15 & -L/30 \\ & & & 2L/15 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

where the kernel matrix is the buckling matrix. A unit axial load of F=1 is assumed here. L is the beam length.

The degrees of freedom are shown in the following sketch.



Subroutine BAR calculates (on option) finite element (1) mass matrix; (2) stiffness matrices (same as global load transformation matrices); (3) buckling matrices for unit load; (4) local load transformation matrices; (5) stress transformation matrices; and (6) vectors to locate the DOF (degrees of freedom) of the above matrices in the global DOF, for combined axial-torsion-bending bar elements. The above matrices and vectors are written on disk units and constitute the output from this subroutine. All matrices are in dense programming logic.

Each mass, stiffness, and buckling matrix, size 12x12, is in the global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1, then joint 2 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U = \delta_X$, $V = \delta_Y$, $W = \delta_Z$, $P = \theta_X$, $Q = \theta_Y$, $R = \theta_Z$.

Each global load transformation matrix, size 12x12, relates loads at the bar ends in the global coordinate directions to deflections in the global coordinate directions. The row order in this matrix is (P_U, P_V, P_W, M_P, M_Q, M_R) joint 1, then joint 2 where P is force and M is moment.

Each local load transformation matrix, size 12x12, relates loads at the bar ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is P_{x1}, P_{x2}, M_{x1}, M_{x2}, P_{y1}, P_{y2}, M_{z1}, M_{z2}, P_{z1}, P_{z2}, M_{y1}, M_{y2}, where P is force and M is moment.

Each stress transformation matrix, size 12x12, relates stresses at the bar ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is

$$\frac{P_{x1}}{A_1}, \quad \frac{P_{x2}}{A_2}, \quad \frac{M_{x1} * r_1}{J_1}, \quad \frac{M_{x2} * r_2}{J_2},$$

$$\frac{P_{y1}}{A_1}, \quad \frac{P_{y2}}{A_2}, \quad \frac{M_{z1} * c_{y1}}{I_{z1}}, \quad \frac{M_{z2} * c_{y2}}{I_{z2}}$$

$$\frac{P_{z1}}{A_1}, \quad \frac{P_{z2}}{A_2}, \quad \frac{M_{y1} * c_{z1}}{I_{y1}}, \quad \frac{M_{y2} * c_{z2}}{I_{y2}}$$

where P is force, M is moment, A is cross-sectional area, r is distance from torsion axis (x), to outer fiber, J is cross-section Saint Venant's torsion constant in JG, c is distance from bending neutral plane to outer fiber, and I is area moment of inertia.

Each location vector (IVEC) locates the DOF of each finite element in the global DOF. For example, IVEC(6)=834 places element DOF 6 into global DOF 834. IVEC(3)=0 omits element DOF 3 from global DOF. This constrains element DOF 3 to zero motion.

These matrices are calculated by using joint data and element data. The joint data is obtained from three matrices input to this subroutine which are (1) joint global X, Y, Z locations; (2) joint global DOF numbers; and (3) joint Euler angles.

The element data are (1) options for mass, stiffness, local load transformations, stress transformations; (2) element material properties, and (3) element joint numbers.

Each mass matrix is calculated by transfer to subroutine MAS1B. Each stiffness matrix, loads, and stress transformation matrix is calculated by transfer to subroutine STF1B. Each unit load buckling matrix is calculated by transfer to subroutine BUC1B.

Subroutine BUC1B calculates a finite element buckling (sometimes referred to as geometrical stiffness, initial stress, or stability) matrix for a combined axial-torsion-bending bar element with unrestrained boundaries. The buckling matrix is based on unit axial load.

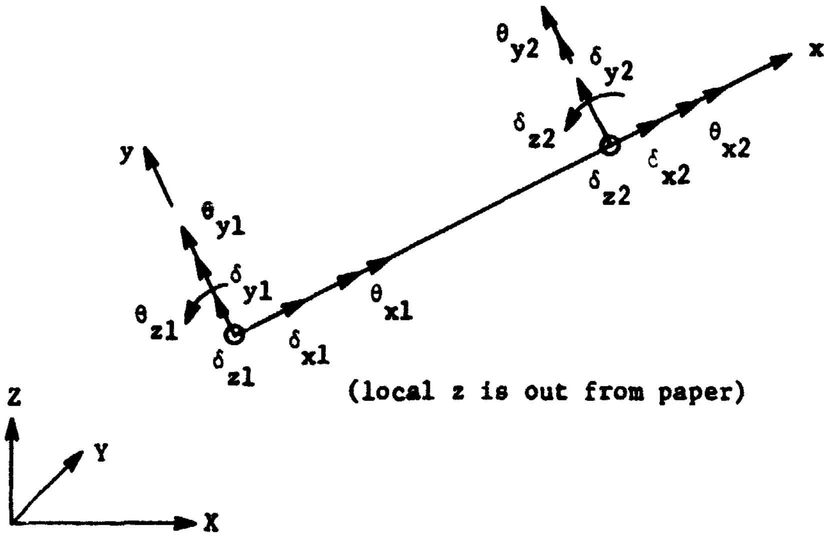
The buckling matrix, size 12x12, is in the global coordinate directions. The global coordinate order for each element is (U,V,W,P,Q,R) joint 1, then joint 2 where U,V,W are translations and P,Q,R are rotations. If the Euler angles are zero at a joint, the $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

This matrix is calculated by first computing a buckling matrix in the local coordinate system for either an axial rod (where the buckling matrix is sometimes referred to as the string stiffness) or a beam. A direction cosine matrix is then used to transform the buckling matrix from the local coordinate system to the global coordinate system.

DESCRIPTION OF TECHNIQUE

The calculation of the buckling matrix in the global coordinate directions is accomplished as follows. First a buckling matrix is calculated in the local coordinate system for either an axial rod (reference Subroutine B1A1) or a beam (reference Subroutine B1A2).

A sketch of the bar is given for reference as



(local z is out from paper)

The strain energy for buckling or geometric stiffness using local coordinates is

$$U = \frac{1}{2} \{h_L\}^T [b_L] \{h_L\} \qquad [1]$$

where

$$\{h_L\}^T = [\delta_{x1} \; \delta_{x2} \; \vdots \; \theta_{x1} \; \theta_{x2} \; \vdots \; \delta_{y1} \; \delta_{y2} \; \theta_{z1} \; \theta_{z2} \; \vdots \; \delta_{z1} \; \delta_{z2} \; \theta_{y1} \; \theta_{y2}]$$

and

$$[b_L] = \begin{bmatrix}
0 & & & & & & & & & & & \\
& 0 & & & & & & & & & & \\
& & 0 & & & & & & & & & \\
& & & 0 & & & & & & & & \\
& & & & b_{11} & b_{12} & -b_{13} & -b_{14} & & & & \\
& & & & b_{21} & b_{22} & -b_{23} & -b_{24} & & & & \\
& & & & -b_{31} & -b_{32} & b_{33} & b_{34} & & & & \\
& & & & -b_{41} & -b_{42} & b_{43} & b_{44} & & & & \\
& & & & & & & & b_{11} & b_{12} & b_{13} & b_{14} \\
& & & & & & & & b_{21} & b_{22} & b_{23} & b_{24} \\
& & & & & & & & b_{31} & b_{32} & b_{33} & b_{34} \\
& & & & & & & & b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix}$$

b_{ij} refers to terms from Subroutines B1A1 or B1A2.

The deflections in the local system are related to deflections in the global system by

$$\{h_L\} = [\lambda] \{h_G\} \qquad [2]$$

where $[\gamma]$ is a direction cosine matrix (reference Subroutine DCOS1B) including Euler angles, size 12x12, and

$$\{h_G\}^T = [U_1\ V_1\ W_1\ P_1\ Q_1\ R_1\ U_2\ V_2\ W_2\ P_2\ Q_2\ R_2].$$

U, V, W are translations and P, Q, R are rotations.

Substituting Eq [2] into Eq [1] gives

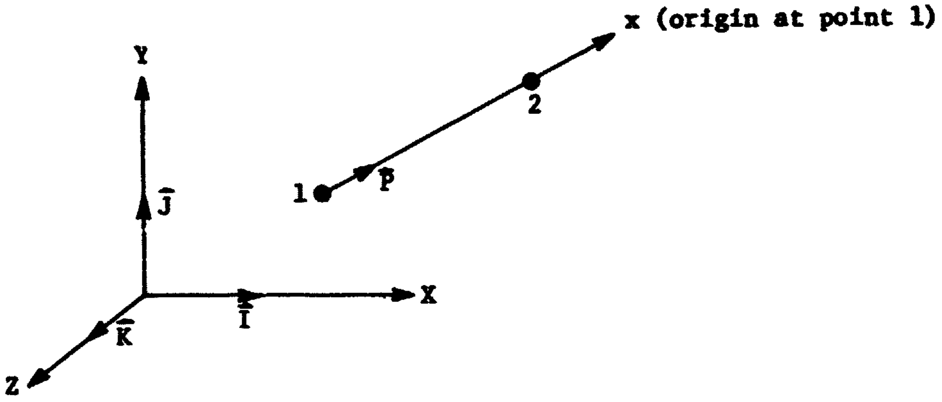
$$U = \frac{1}{2}\{h_G\}^T [b_G] \{h_G\} \qquad [3]$$

where $[b_G] = [\gamma]^T [b_L] [\gamma]$ is the buckling matrix in global coordinate directions.

Subroutine DCOS1A calculates a direction cosine matrix for an axial rod element. This matrix relates local coordinate displacements to global coordinate displacements. Euler angles at each joint are included. Global X, Y, Z coordinates and Euler angles at each of the two rod ends are needed for this calculation.

DESCRIPTION OF TECHNIQUE

A sketch of the rod is given for reference as



The vector $\vec{P}$ between points 1 and 2 is

$$\vec{P} = P_X \vec{I} + P_Y \vec{J} + P_Z \vec{K} \qquad [1]$$

The unit vector is then

$$\vec{e}_P = [P_X \vec{I} + P_Y \vec{J} + P_Z \vec{K}]/\ell_P \qquad [2]$$

where

$$P_X = X_2 - X_1$$

$$P_Y = Y_2 - Y_1$$

$$P_Z = Z_2 - Z_1$$

$$\ell_P = \sqrt{P_X^2 + P_Y^2 + P_Z^2}\ .$$

The coefficients of $\vec{I}$, $\vec{J}$, $\vec{K}$ for the unit vector $\vec{e}_p$ are the direction cosines of the line between points 1 and 2.

The relation between local and global X, Y, Z displacements is then

$$\begin{bmatrix} \delta_{x1} \\ \delta_{x2} \end{bmatrix} = \begin{bmatrix} [e_p] & 0 \\ 0 & [e_p] \end{bmatrix} \begin{bmatrix} \{\delta_{XYZ}\}_1 \\ \{\delta_{XYZ}\}_2 \end{bmatrix} \qquad [3]$$

where

$$\{\delta_{XYZ}\} = \begin{bmatrix} \delta_X \\ \delta_Y \\ \delta_Z \end{bmatrix}.$$

A 3x3 Euler angle transformation matrix (reference subroutine EULER) relates global X, Y, Z translations to global U, V, W translations at each joint. That is,

$$\begin{bmatrix} \{\delta_{XYZ}\}_1 \\ \{\delta_{XYZ}\}_2 \end{bmatrix} = \begin{bmatrix} [E]_1 & 0 \\ 0 & [E]_2 \end{bmatrix} \begin{bmatrix} \{UVW\}_1 \\ \{UVW\}_2 \end{bmatrix} \qquad [4]$$

where

$$\{UVW\} = \begin{bmatrix} U \\ V \\ W \end{bmatrix}.$$

Substituting Eq [4] into Eq [3] gives the direction cosine matrix.

$$\begin{bmatrix} \delta_{x1} \\ \delta_{x2} \end{bmatrix} = \begin{bmatrix} [e_p]_1 & 0 \\ 0 & [e_p]_2 \end{bmatrix} \begin{bmatrix} \{UVW\}_1 \\ \{UVW\}_2 \end{bmatrix} \qquad [5]$$

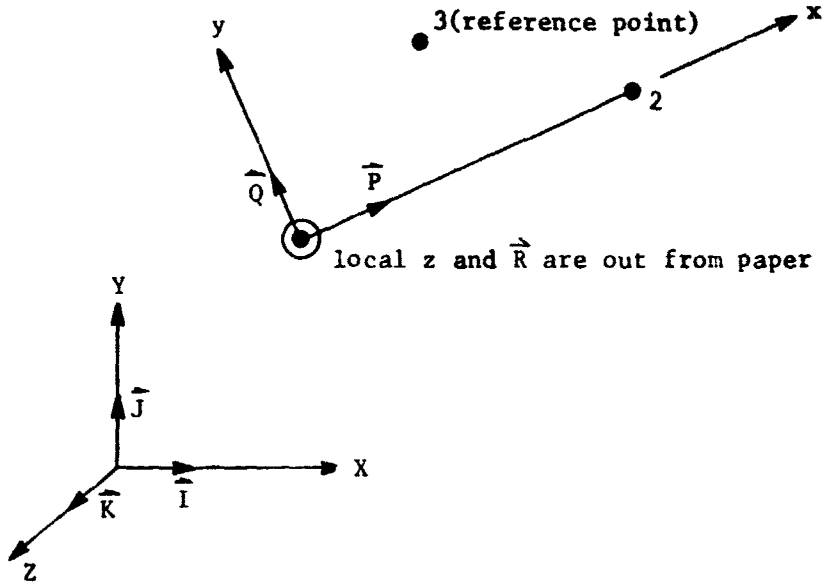
where

$[e_p]_1 = [e_p]\ [E]_1$, etc.

Subroutine DCOS1B calculates a direction cosine matrix for a combined axial-torsion-bending bar element. This matrix relates local coordinate displacements to global coordinate displacements. Euler angles at each joint are included. Global X, Y, Z coordinates and Euler angles at each of the two bar ends plus coordinates of a reference point are needed for this calculation. The reference point defines the local xy plane.

DESCRIPTION OF TECHNIQUE

A sketch of the bar is given for reference as



The vector $\vec{P}$ between points 1 and 2 is

$$\vec{P} = P_X \vec{I} + P_Y \vec{J} + P_Z \vec{K} \quad [1]$$

The unit vector is then

$$\vec{e}_P = [P_X \vec{I} + P_Y \vec{J} + P_Z \vec{K}]/L_P \quad [2]$$

where

$P_X = X_2 - X_1$

$P_Y = Y_2 - Y_1$

$P_Z = Z_2 - Z_1$

$\ell_P = \sqrt{P_X^2 + P_Y^2 + P_Z^2}$

The coefficients of $\overrightarrow{I}$, $\overrightarrow{J}$, $\overrightarrow{K}$ for the unit vector $\overrightarrow{e}_P$ are the direction cosines of the line between points 1 and 2.

The vector cross poduct $\overrightarrow{P} \times \overrightarrow{1,\ 3}$ will give a vector $(\overrightarrow{R}) \perp$ plane 1, 2, 3.

$$\overrightarrow{R} = \overrightarrow{P} \times \overrightarrow{1,3} = [P_X\ \overrightarrow{I} + P_Y\ \overrightarrow{J} + P_Z\ \overrightarrow{K}] \times [(X_3-X_1)\ \overrightarrow{I} + (Y_3-Y_1)\ \overrightarrow{J} + (Z_3-Z_1)\ \overrightarrow{K}]$$
$$= R_X\ \overrightarrow{I} + R_Y\ \overrightarrow{J} + R_Z\ \overrightarrow{K}. \qquad [3]$$

The unit vector along R is then

$$\overrightarrow{e}_R = [R_X\ \overrightarrow{I} + R_Y\ \overrightarrow{J} + R_Z\ \overrightarrow{K}]/\ell_R \qquad [4]$$

where

$R_X = P_Y\ (Z_3-Z_1) - P_Z\ (Y_3-Y_1)$

$R_Y = P_Z\ (X_3-X_1) - P_X\ (Z_3-Z_1)$

$R_Z = P_X\ (Y_3-Y_1) - P_Y\ (X_3-X_1)$

$\ell_R = \sqrt{R_X^2 + R_Y^2 + R_Z^2}$

The coefficients of $\overrightarrow{I}$, $\overrightarrow{J}$, $\overrightarrow{K}$ for the unit vector $\overrightarrow{e}_R$ are the direction cosines of a line $\perp$ plane 1, 2, 3.

The vector cross product $\overrightarrow{R} \times \overrightarrow{P}$ will give a vector $(\overrightarrow{Q}) \perp$ line 1, 2 in the plane 1, 2, 3.

$$\overrightarrow{Q} = \overrightarrow{R} \times \overrightarrow{P} = [R_X\ \overrightarrow{I} + R_X\ \overrightarrow{J} + R_Z\ \overrightarrow{K}] \times [P_X\ \overrightarrow{I} + P_Y\ \overrightarrow{J} + P_Z\ \overrightarrow{K}] \qquad [5]$$
$$= Q_X\ \overrightarrow{I} + Q_Y\ \overrightarrow{J} + Q_Z\ \overrightarrow{K}] .$$

The unit vector along $\vec{Q}$ is then

$$\vec{e}_Q = \frac{1}{\ell_Q} [Q_X \vec{I} + Q_Y \vec{J} + Q_Z \vec{K}] \qquad [6]$$

where

$$Q_X = R_Y P_Z - R_Z P_Y$$

$$Q_Y = R_Z P_X - R_X P_Z$$

$$Q_Z = R_X P_Y - R_Y P_X$$

$$\ell_Q = \sqrt{Q_X^2 + Q_Y^2 + Q_Z^2}$$

The coefficients of $\vec{I}$, $\vec{J}$, $\vec{K}$ for the unit vector $\vec{e}_Q$ are the direction cosines of a $\perp$ line 1, 2 and in the plane 1, 2, 3.

The relation between local and global X, Y, Z displacements is then

$$\begin{bmatrix} \delta_{x1} \\ \delta_{x2} \\ \theta_{x1} \\ \theta_{x2} \\ \delta_{y1} \\ \delta_{y2} \\ \theta_{z1} \\ \theta_{z2} \\ \delta_{z1} \\ \delta_{z2} \\ \theta_{y1} \\ \theta_{y2} \end{bmatrix} \begin{bmatrix} [e_P] & & & \\ & & [e_P] & \\ & [e_P] & & \\ & & & [e_P] \\ [e_Q] & & & \\ & & [e_Q] & \\ & [e_R] & & \\ & & & [e_R] \\ [e_R] & & & \\ & & [e_R] & \\ & [e_Q] & & \\ & & & [e_Q] \end{bmatrix} \begin{bmatrix} \{\delta_{XYZ}\}_1 \\ \{\theta_{XYZ}\}_1 \\ \{\delta_{XYZ}\}_2 \\ \{\theta_{XYZ}\}_2 \end{bmatrix} .$$

where $\{\delta_{XYZ}\} = \begin{bmatrix}\delta_X\\ \delta_Y\\ \delta_Z\end{bmatrix}$ and $\{\theta_{XYZ}\} = \begin{bmatrix}\theta_X\\ \theta_Y\\ \theta_Z\end{bmatrix}$. [7]

A 3x3 Euler angle transformation matrix (reference subroutine EULER) relates global X,Y,Z displacements to global U,V,W translations and P,Q,R rotations. That is,

$$\begin{bmatrix}\{\delta_{XYZ}\}_1\\ \{\theta_{XYZ}\}_1\\ \{\delta_{XYZ}\}_2\\ \{\theta_{XYZ}\}_2\end{bmatrix} = \begin{bmatrix}[E]_1 & & & \\ & [E]_1 & & \\ & & [E]_2 & \\ & & & [E]_2\end{bmatrix} \begin{bmatrix}\{UVW\}_1\\ \{PQR\}_1\\ \{UVW\}_2\\ \{PQR\}_2\end{bmatrix} \qquad [8]$$

where $\{UVW\} = \begin{bmatrix}U\\ V\\ W\end{bmatrix}$ and $\{PQR\} = \begin{bmatrix}P\\ Q\\ R\end{bmatrix}$.

Substituting Eq [8] into Eq [7] gives the direction cosine matrix.

$$\begin{bmatrix}\delta_{x1}\\ \delta_{x2}\\ \theta_{x1}\\ \theta_{x2}\\ \delta_{y1}\\ \delta_{y2}\\ \theta_{z1}\\ \theta_{z2}\\ \delta_{z1}\\ \delta_{z2}\\ \theta_{y1}\\ \theta_{y2}\end{bmatrix} \begin{bmatrix}[e_P]_1 & & & \\ & & [e_P]_2 & \\ & [e_P]_1 & & \\ & & & [e_P]_2\\ [e_Q]_1 & & & \\ & & [e_Q]_2 & \\ & [e_R]_1 & & \\ & & & [e_R]_2\\ [e_R]_1 & & & \\ & & [e_R]_2 & \\ & [e_Q]_1 & & \\ & & & [e_Q]_2\end{bmatrix} \begin{bmatrix}\{UVW\}_1\\ \{PQR\}_1\\ \{UVW\}_2\\ \{PQR\}_2\end{bmatrix} .$$

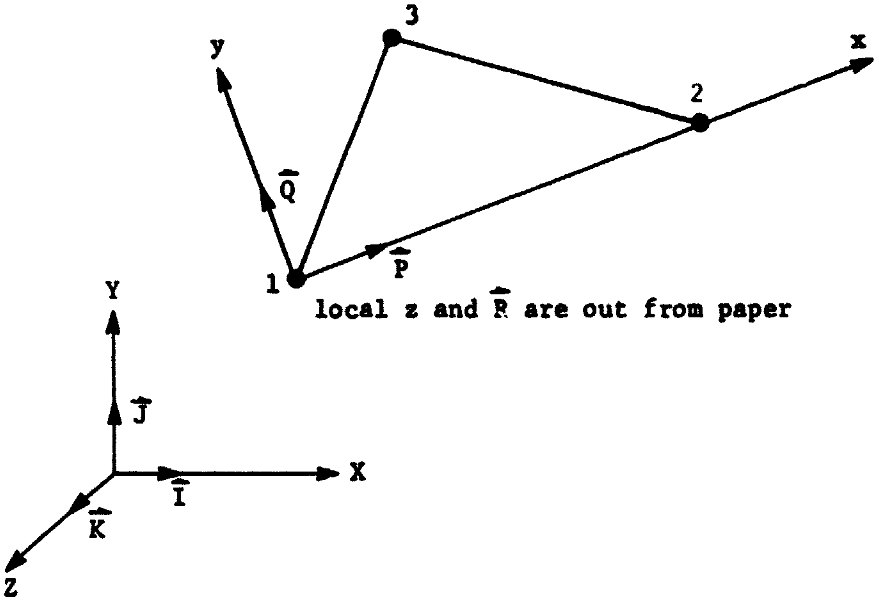
where

$[e_p]_1 = [e_p]\ [E]_1$ etc.

Subroutine DCOS2 calculates a direction cosine matrix for a combined membrane-bending triangle plate element. This matrix relates local coordinate displacements to global coordinate displacements. Euler angles at each joint are included. Global X, Y, Z coordinates and Euler angles at each of the three corners are needed for this calculation.

DESCRIPTION OF TECHNIQUE

A sketch of the triangle plate is given for reference as



local z and $\vec{R}$ are out from paper

The vector P between points 1 and 2 is

$$\vec{P} = P_X \vec{I} + P_Y \vec{J} + P_Z \vec{K} \quad [1]$$

The unit vector is then

$$\vec{e}_P = [P_X \vec{I} + P_Y \vec{J} + P_Z \vec{K}]/\ell_P \quad [2]$$

where

$P_X = X_2 - X_1$

$P_Y = Y_2 - Y_1$

$P_Z = Z_2 - Z_1$

$\ell_P = \sqrt{P_X^2 + P_Y^2 + P_Z^2}$

The coefficients of $\overrightarrow{I}$, $\overrightarrow{J}$, $\overrightarrow{K}$ for the unit vector $\overrightarrow{e_P}$ are the direction cosines of the line between points 1 and 2.

The vector cross product $\overrightarrow{P} \times \overrightarrow{1, 3}$ will give a vector $(\overrightarrow{R}) \perp$ plane 1, 2, 3.

$$\overrightarrow{R} = \overrightarrow{P} \times \overrightarrow{1,3} = [P_X \overrightarrow{I} + P_Y \overrightarrow{J} + P_Z \overrightarrow{K}] \times [(X_3-X_1)\overrightarrow{I} + (Y_3-Y_1) J + (Z_3-Z_1) K]$$

$$= R_X \overrightarrow{I} + R_Y \overrightarrow{J} + R_Z \overrightarrow{K}. \qquad [3]$$

The unit vector along $\overrightarrow{R}$ is then

$$\overrightarrow{e}_R = [R_X \overrightarrow{I} + R_Y \overrightarrow{J} + R_Z \overrightarrow{K}]/\ell_R \qquad [4]$$

where

$R_X = P_Y (Z_3-Z_1) - P_Z (Y_3-Y_1)$

$R_Y = P_Z (X_3-X_1) - P_X (Z_3-Z_1)$

$R_Z = P_X (Y_3-Y_1) - P_Y (X_3-X_1)$

$\ell_R = \sqrt{R_X^2 + R_Y^2 + R_Z^2}$

The coefficients of $\overrightarrow{I}$, $\overrightarrow{J}$, $\overrightarrow{K}$ for the unit vector e_R are the direction cosines of a line $\perp$ plane 1, 2, 3.

The vector cross product $\overrightarrow{R} \times \overrightarrow{P}$ will give a vector $(\overrightarrow{Q}) \perp$ line 1, 2 in the plane 1, 2, 3.

$$\begin{aligned} \overrightarrow{Q} = \overrightarrow{R} \times \overrightarrow{P} &= [R_X \overrightarrow{I} + R_Y \overrightarrow{J} + R_Z \overrightarrow{K}] \times [P_X \overrightarrow{I} + P_Y \overrightarrow{J} + P_Z \overrightarrow{K}] \\ &= Q_X \overrightarrow{I} + Q_Y \overrightarrow{J} + Q_Z \overrightarrow{K}. \end{aligned} \qquad [5]$$

The unit vector along R is then

$$\overrightarrow{e_Q} = \frac{1}{\ell_Q} \, [Q_X \overrightarrow{I} + Q_Y \overrightarrow{J} + Q_Z \overrightarrow{K}] \qquad [6]$$

where

$$Q_X = R_Y P_Z - R_Z P_Y$$

$$Q_Y = R_Z P_X - R_X P_Z$$

$$Q_Z = R_X P_Y - R_Y P_X$$

$$\ell_Q = \sqrt{Q_X^2 + Q_Y^2 + Q_Z^2} \quad .$$

The coefficients of $\overrightarrow{I}$, $\overrightarrow{J}$, $\overrightarrow{K}$ for the unit vector e_Q are the direction cosines of a line $\perp$ line 1,2 and in the plane 1, 2, 3.

The relation between local and global X, Y, Z displacements is then

$$\left[\begin{array}{c} \begin{Bmatrix}\delta_x\\ \delta_y\\ \theta_z\end{Bmatrix}_1 \\ \begin{Bmatrix}\delta_x\\ \delta_y\\ \theta_z\end{Bmatrix}_2 \\ \begin{Bmatrix}\delta_x\\ \delta_y\\ \theta_z\end{Bmatrix}_3 \\ \hline \begin{Bmatrix}\delta_z\\ \theta_x\\ \theta_y\end{Bmatrix}_1 \\ \begin{Bmatrix}\delta_z\\ \theta_x\\ \theta_y\end{Bmatrix}_2 \\ \begin{Bmatrix}\delta_z\\ \theta_x\\ \theta_y\end{Bmatrix}_3 \end{array}\right] \left[\begin{array}{c|c|c|c|c|c} [e_P] & & & & & \\ [e_Q] & & & & & \\ & [e_R] & & & & \\ \hline & & [e_P] & & & \\ & & [e_Q] & & & \\ & & & [e_R] & & \\ \hline & & & & [e_P] & \\ & & & & [e_Q] & \\ & & & & & [e_R] \\ \hline [e_R] & & & & & \\ & [e_P] & & & & \\ & [e_Q] & & & & \\ \hline & & [e_R] & & & \\ & & & [e_P] & & \\ & & & [e_Q] & & \\ \hline & & & & [e_R] & \\ & & & & & [e_P] \\ & & & & & [e_Q] \end{array}\right] \begin{bmatrix} \{\delta_{XYZ}\}_1 \\ \{\theta_{XYZ}\}_1 \\ \{\delta_{XYZ}\}_2 \\ \{\theta_{XYZ}\}_2 \\ \{\delta_{XYZ}\}_3 \\ \{\theta_{XYZ}\}_3 \end{bmatrix} \tag*{[7]}$$

where $\{\delta_{XYZ}\} = \begin{bmatrix} \delta_X \\ \delta_Y \\ \delta_Z \end{bmatrix}$ and $\{\theta_{XYZ}\} = \begin{bmatrix} \theta_X \\ \theta_Y \\ \theta_Z \end{bmatrix}$.

A 3x3 Euler angle transformation matrix (reference subroutine EULER) relates global X, Y, Z displacements to global U, V, W translations and P, Q, R rotations. That is,

$$\begin{bmatrix} \{\delta_{XYZ}\}_1 \\ \{\theta_{XYZ}\}_1 \\ \{\delta_{XYZ}\}_2 \\ \{\theta_{XYZ}\}_2 \\ \{\delta_{XYZ}\}_3 \\ \{\theta_{XYZ}\}_3 \end{bmatrix} = \begin{bmatrix} [E]_1 & & & & & \\ & [E]_1 & & & & \\ & & [E]_2 & & & \\ & & & [E]_2 & & \\ & & & & [E]_3 & \\ & & & & & [E]_3 \end{bmatrix} \begin{bmatrix} \{UVW\}_1 \\ \{PQR\}_1 \\ \{UVW\}_2 \\ \{PQR\}_2 \\ \{UVW\}_3 \\ \{PQR\}_3 \end{bmatrix} \qquad [8]$$

where $\{UVW\} = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$ and $\{PQR\} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$.

Substituting Eq [8] into Eq [7] gives the direction cosine matrix.

$$\left[\begin{array}{c} \begin{Bmatrix}\delta_x\\ \delta_y\\ \theta_z\end{Bmatrix}_1 \\ \begin{Bmatrix}\delta_x\\ \delta_y\\ \theta_z\end{Bmatrix}_2 \\ \begin{Bmatrix}\delta_x\\ \delta_y\\ \theta_z\end{Bmatrix}_3 \\ \hline \begin{Bmatrix}\delta_z\\ \theta_x\\ \theta_y\end{Bmatrix}_1 \\ \begin{Bmatrix}\delta_z\\ \theta_x\\ \theta_y\end{Bmatrix}_2 \\ \begin{Bmatrix}\delta_z\\ \theta_x\\ \theta_y\end{Bmatrix}_3 \end{array}\right] \left[\begin{array}{c|c|c|c|c|c} [e_P]_1 & & & & & \\ [e_Q]_1 & & & & & \\ & [e_R]_1 & & & & \\ \hline & & [e_P]_2 & & & \\ & & [e_Q]_2 & & & \\ & & & [e_R]_2 & & \\ \hline & & & & [e_P]_3 & \\ & & & & [e_Q]_3 & \\ & & & & & [e_R]_3 \\ \hline [e_R]_1 & & & & & \\ & [e_P]_1 & & & & \\ & [e_Q]_1 & & & & \\ \hline & & [e_R]_2 & & & \\ & & & [e_P]_2 & & \\ & & & [e_Q]_2 & & \\ \hline & & & & [e_R]_3 & \\ & & & & & [e_P]_3 \\ & & & & & [e_Q]_3 \end{array}\right] \begin{bmatrix} \{UVW\}_1 \\ \{PQR\}_1 \\ \{UVW\}_2 \\ \{PQR\}_2 \\ \{UVW\}_3 \\ \{PQR\}_3 \end{bmatrix}$$

Where $[e_P]_1 = [e_P]\,[E]_1$, etc.

Subroutine EULER calculates the Euler rotation transformation matrix such that

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = [T] \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}$$

where

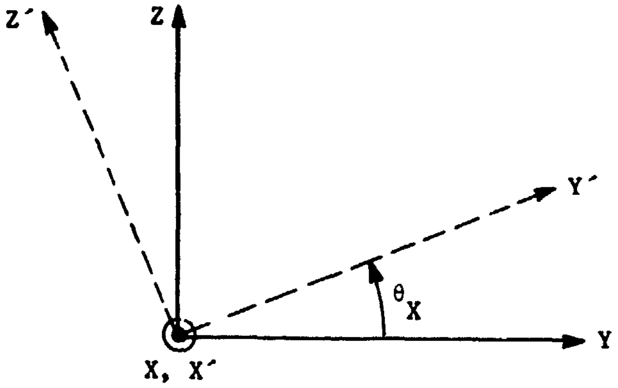
$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$ = global X, Y, Z coordinate system.

$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix}$ = rotated U, V, W coordinate system.

[T] = Euler rotation transformation based on a global θ_X, θ_Y, and θ_Z permutation.

DESCRIPTION OF TECHNIQUE

The first Euler rotation is θ_X about X to form the X′, Y′, Z′ coordinate system.



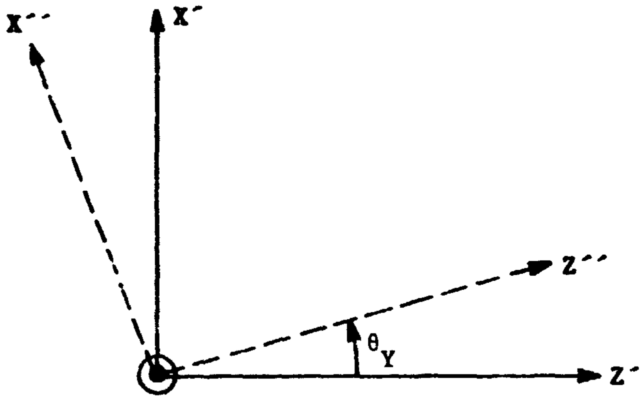
The relationship between the two coordinate systems can be written as

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = [T\theta_X] \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix}$$

where

$$[T\theta_X] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & -\sin\theta_X \\ 0 & \sin\theta_X & \cos\theta_X \end{bmatrix}$$

The second Euler rotation is θ_Y about Y' to form the X'', Y'', Z'' coordinate system.



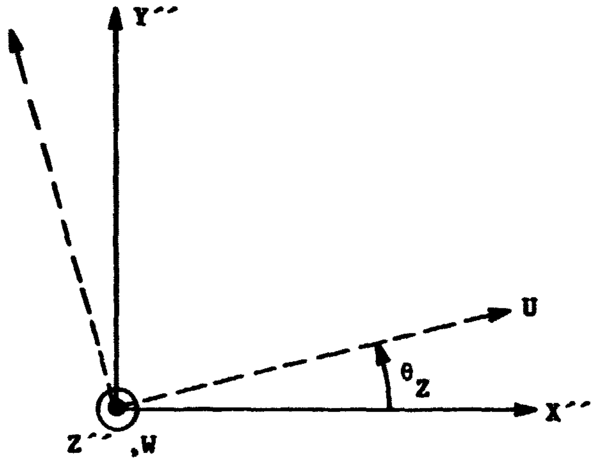
The relationship between the two coordinate system can be written as

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = [T\theta_Y] \begin{Bmatrix} X'' \\ Y'' \\ Z'' \end{Bmatrix}$$

where

$$[T\theta_Y] = \begin{bmatrix} \cos\theta_Y & 0 & \sin\theta_Y \\ 0 & 1 & 0 \\ -\sin\theta_Y & 0 & \cos\theta_Y \end{bmatrix}$$

The third Euler rotation is θ_Z about Z'' to form the U, V, W coordinate system.



The relationship between the two coordinate systems can be written as

$$\begin{Bmatrix} X'' \\ Y'' \\ Z'' \end{Bmatrix} = [T\theta_Z] \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}$$

where

$$[T\theta_Z] \begin{bmatrix} \cos\theta_Z & -\sin\theta_Z & 0 \\ \sin\theta_Z & \cos\theta_Z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The complete Euler rotation transformation can be written as

$$[T] = [T\theta_X]\ [T\theta_Y]\ [T\theta_Z]$$

Performing the three multiplications results in

$$[T] = \left[\begin{array}{l|l|l}
\cos\theta_Y \ \cos\theta_Z & -\cos\theta_Y \ \sin\theta_Z & \sin\theta_Y \\
\hline
\begin{array}{l}\cos\theta_X \ \sin\theta_Z \\ +\sin\theta_X \ \sin\theta_Y \ \cos\theta_Z\end{array} & \begin{array}{l}\cos\theta_X \ \cos\theta_Z \\ -\sin\theta_X \ \sin\theta_Y \ \sin\theta_Z\end{array} & -\sin\theta_X \ \cos\theta_Y \\
\hline
\begin{array}{l}\sin\theta_X \ \sin\theta_Z \\ -\cos\theta_X \ \sin\theta_Y \ \cos\theta_Z\end{array} & \begin{array}{l}\sin\theta_X \ \cos\theta_Z \\ +\cos\theta_X \ \sin\theta_Y \ \sin\theta_Z\end{array} & \cos\theta_X \ \cos\theta_Y
\end{array}\right]$$

Subroutine FINEL calculates (on option) using finite elements: 1) an assembled mass matrix; 2) an assembled stiffness matrix; 3) element local load transformation matrices; 4) element global load transformation matrices; 5) element stress transformation matrices; 6) element unit load buckling matrices; and 7) vectors (IVEC) to locate the DOF (degree of freedom) of the element matrices in the global DOF.

The types of finite element available (and the related subroutine) are axial rod (AXIAL), combined axial-torsion-bending bar (BAR), triangular plate (TRNGL), quadrilateral plate (QUAD), rectangular shear panel (RECTSP), tetrahedron (TETRA), and pentahedron (PENTA). The subroutine to be used is specified by reading this information (e.g., AXIAL, BAR, etc.) from an input data card.

The assembled mass and stiffness matrices are output from this subroutine in sparse FORMA subroutine format on disk units. The DOF order is specified by a joint degree-of-freedom matrix, [JDOF] , which is input to this subroutine.

The element matrices and vectors are in dense programming logic and written on disk units as output from this subroutine also. The sizes of these element matrices and vectors are determined by the specific finite element used. Each vector (IVEC) locates the DOF of each finite element in the global DOF. For example, IVEC(6)=834 places element DOF 6 into global DOF 834. IVEC(3)=0 omits element DOF 3 from global DOF. This constrains element DOF 3 to zero motion.

The finite element matrices are calculated by using joint data and element data. The joint data, obtained from three matrices input to this subroutine, are 1) joint global X, Y, Z locations; 2) joint global DOF numbers; and 3) joint Euler angles. The element data is read in the specified finite element subroutine. Reference AXIAL, BAR, etc. for this data.

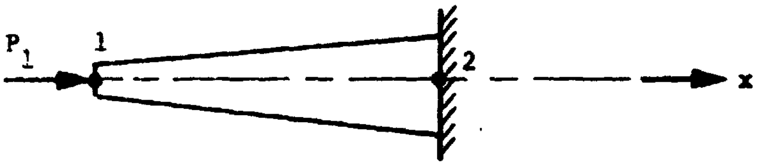
Assembly of the element mass (or stiffness) matrices into the assembled mass (or stiffness) matrix for the total structure is accomplished by FORMA subroutine YRVAD2 to obtain the sparse subroutine format.

Subroutine K1A1 calculates a stiffness matrix and stress transformation matrix for an axial rod element with unrestrained boundaries. The stiffness matrix is in the local coordinate system of the rod. The elements of the stiffness matrix represent the distributed stiffness properties of the rod. These elements are calculated by assuming constant axial force. The stress transformation matrix relates stress at the rod ends in the local coordinate system to deflections in the local coordinate system. The rod may be linearly tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed axial stiffness of a rod by a stiffness matrix is obtained using a strain energy approach as follows.

Consider a rod that is loaded with an axial force P_1 at point 1 and restrained at point 2 as shown in the sketch.



Rod Element

The strain energy is defined by

$$U = \frac{1}{2}\int_{x_1}^{x_2} \frac{P(x)^2}{A(x)\ E(x)}\ dx \qquad [1]$$

Where

P is the axial force,

A is the cross-sectional area,

E is Young's modulus of elasticity

x is the local coordinate system and longitudinal axis of the rod. The origin is at point 1, that is, $x_1 = 0$; $x_2 = L$ (rod length).

To integrate Eq [1], the axial force is assumed constant and equal to the axial force at point 1, that is,

$$P(x) = P_1. \qquad [2]$$

Young's modulus of elasticity is also assumed constant, that is,

$$E(x) = E. \qquad [3]$$

The cross-sectional area is assumed to vary linearly, that is,

$$A(x) = A_1 + x\,(A_2 - A_1)/L. \qquad [4]$$

Substituting Eq [2] through [4] into Eq [1] gives the strain energy as

$$U = \frac{1}{2}\,\frac{P_1^2}{E}\int_0^L \frac{1}{A_1 + x\,(A_2-A_1)/L}\,dx \qquad [5]$$

$$= \frac{1}{2}\,\frac{P_1^2\,L}{(A_2-A_1)\,E}\,\ell n\,(A_2/A_1) \qquad [6]$$

Application of Castigliano's theorem gives the axial deflection of point 1 relative to point 2 as

$$\Delta\delta = \frac{\partial U}{\partial P_1} = \frac{P_1 L}{(A_2-A_1)\,E}\,\ell n\,(A_2/A_1)$$

from which

$$P_1 = \frac{(A_2-A_1)\,E}{L\,\ell n\,(A_2/A_1)}\,\Delta\delta \qquad [7]$$

The restraint at point 2 is removed by application of the transformation

$$\Delta\delta = [1 \quad -1] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \qquad [8]$$

where δ_1 and δ_2 are the displacements along the rod x-axis at rod-ends 1 and 2, respectively. Substitution of Eq [7] and Eq [8] into Eq [6] gives the strain energy for a rod with unrestrained boundaries as

$$U = \frac{1}{2} \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix} \begin{bmatrix} z_{1,1} & z_{1,2} \\ \text{(sym)} & z_{2,2} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \qquad [9]$$

where

$$z_{1,1} = \frac{(A_2-A_1)\ E}{L\ \ln\ (A_2/A_1)} \qquad [10]$$

$$z_{1,2} = -\ z_{1,1} \qquad [11]$$

$$z_{2,2} = z_{1,1} \qquad [12]$$

The kernel matrix of Eq [9] is the stiffness matrix that represents the axial stiffness of a rod with unrestrained boundaries.

For constant cross-sectional area, i.e., $A_1 = A_2 = A$, Eq [10] is of indefinite form. For this case integration of Eq [5] yields

$$U = \frac{1}{2} \frac{P_1^2\ L}{AE} \qquad [6a]$$

from which

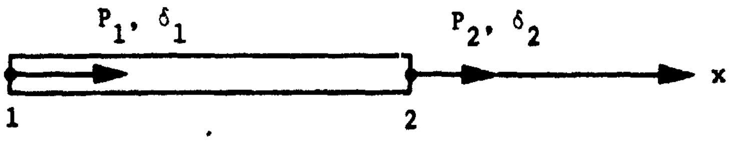
$$z_{1,1} = \frac{AE}{L}\ . \qquad [10a]$$

as before

$$z_{1,2} = -\ z_{1,1}$$

$$z_{2,2} = z_{1,1}$$

The elements in the stress transformation matrix are easily calculated. The following sketch shows the sign convention.



The rod end forces (P_1, P_2) can be expressed in terms of the rod end displacements (δ_1, δ_2) as

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} z_{1,1} & z_{1,2} \\ (\text{sym}) & z_{2,1} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \qquad [13]$$

This is obtained from Eq [7] or applying Castigliano's theorem to Eq [9]. The stress at the rod ends is simply

$$s_1 = P_1/A_1$$

and

$$s_2 = P_2/A_2 \ .$$

or

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = [T_s] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \qquad [14]$$

where

$$[T_s] = \begin{bmatrix} z_{1,1}/A_1 & z_{1,2}/A_1 \\ z_{2,1}/A_2 & z_{2,2}/A_2 \end{bmatrix} \qquad [15]$$

is the stress transformation matrix.

s_1, s_2 will be opposite in sign. Tension and compression in the rod is determined as follows:

Tension: s_1 (-), s_2 (+)

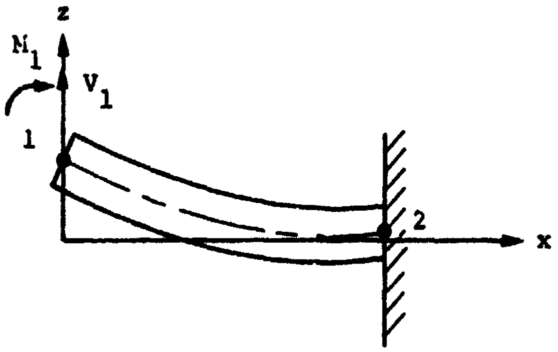
Compression: s_1 (+), s_2 (-)

Subroutine K1B1 calculates a stiffness matrix and stress transformation matrix for a bending (plus shear) beam element with unrestrained boundaries. The stiffness matrix is in the local coordinate system of the beam. The elements of the stiffness matrix represent the distributed stiffness properties of the beam. These elements are calculated by assuming uniform shear and linear bending moment variation. The stress transformation matrix relates stress at the beam ends in the local coordinate system to deflections in the local coordinate system. The beam may be tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed bending and shear stiffness of a beam by a stiffness matrix is obtained using a strain energy approach as follows:

Consider a beam that is loaded with a shear and moment at point 1 and restrained at point 2 as shown in the sketch



The strain energy is defined by

$$U = \frac{1}{2}\int_{x_1}^{x_2} \left(\frac{M(x)^2}{E(x)\ I(x)} + \frac{V(x)^2}{KA(x)\ G(x)} \right) dx \qquad [1]$$

where

$M(x)$	is the bending moment,
$V(x)$	is the shear,
$E(x)$	is Young's modulus of elasticity of the material,
$I(x)$	is the cross-sectional moment of inertia about the beam's neutral axis,
K	is the shape factor (e.g., $K = 1$ for a solid circular cylinder, $K = 0.5$ for a thin walled circular cylinder),
$A(x)$	is the cross-sectional area,
$G(x)$	is the shear modulus of elasticity of the material, and
x	is the local coordinate system and undeformed longitudinal axis of the beam. The origin is at point 1. That is $x_1 = 0$; $x_2 = L$ (rod length).

To integrate Eq [1], the following assumptions are made. First, the shear is assumed *constant* and equal to the shear force at point 1, that is,

$$V(x) = V_1. \qquad [2]$$

Second, the bending moment is assumed to *vary linearly*, that is,

$$M(x) = M_1 + V_1\, x \qquad [3]$$

Third,

$I(x)$ and $A(x)$ are assumed to vary linearly, that is

$$I(x) = I_1 + x\,(I_2 - I_1)/L \qquad [4a]$$

and

$$A(x) = A_1 + x\,(A_2 - A_1)/L \qquad [4b]$$

Fourth, the moduli of elasticity, E and G, are assumed constant, that is

$$E(x) = E \qquad [5a]$$

$$G(x) = G \qquad [5b]$$

Substituting Eq [2] through [5] into Eq [1] gives the strain energy as

$$U = \frac{1}{2}\,[V_1\ M_1] \int_0^L \left(\frac{1}{E\{I_1 + x\,(I_2-I_1)/L\}} \begin{bmatrix} x^2 & x \\ x & 1 \end{bmatrix} + \frac{1}{KG\{A_1 + x\,(A_2-A_1)/L\}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) dx \begin{bmatrix} V_1 \\ M_1 \end{bmatrix} \qquad [6]$$

$$= \frac{1}{2}\,[V_1\ M_1] \begin{bmatrix} f_{11} & f_{12} \\ (\mathrm{sym}) & f_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ M_1 \end{bmatrix} \qquad [7]$$

where

$$f_{11} = \frac{L^3}{EI_1\,(R-1)} \left[\frac{1}{2} - \frac{1}{R-1} + \frac{1}{(R-1)^2}\,\ln R \right] + \frac{L}{KG(A_2-A_1)}\,\ln \frac{A_2}{A_1} \qquad [7a]$$

$$f_{12} = \frac{L^2}{EI_1\,(R-1)} \left[1 - \frac{1}{R-1}\,\ln R \right] \qquad [7b]$$

$$f_{22} = \frac{L}{EI_1\,(R-1)}\,\ln R \qquad [7c]$$

$$R = I_2/I_1 \qquad [7d]$$

For constant bending stiffness, i.e., $EI_1 = EI_2 = EI$, and constant shear stiffness, i.e., $KA_1G = KA_2G = KAG$, Eq [7a] [7b] [7c] are of indefinite form. For this case, integration of Eq [6] yields

$$f_{11} = L^3/3EI + L/KAG \qquad [8a]$$

$$f_{12} = L^2/2EI \qquad [8b]$$

$$f_{22} = L/EI \qquad [8c]$$

Application of Castigliano's theorem to Eq [7] gives the lateral translation and rotation of point 1 relative to point 2 as

$$\begin{bmatrix} \Delta\delta \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} \dfrac{\partial U}{\partial V_1} \\ \dfrac{\partial U}{\partial M_1} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ (\mathrm{sym}) & f_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ M_1 \end{bmatrix} \qquad [9]$$

Solving Eq [9] for V_1 and M_1 and substituting into Eq [7] gives the strain energy as

$$U = \frac{1}{2}\,[\Delta\delta \quad \Delta\theta] \begin{bmatrix} f_{22}/D & -f_{12}/D \\ (\text{sym}) & f_{11}/D \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\theta \end{bmatrix} \qquad [10]$$

where

$D = f_{11}\, f_{22} - f_{12}^2.$

The restraint a point 2 is removed by application of the transformation

$$\begin{bmatrix} \Delta\delta \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -L \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} \qquad [11]$$

Using this transformation in Eq [10] gives the final strain energy expression as

$$U = \frac{1}{2}\,[\text{same as column}] \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ & z_{22} & z_{23} & z_{24} \\ (\text{sym}) & & z_{33} & z_{34} \\ & & & z_{44} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} \qquad [12]$$

$z_{11} = f_{22}/D$

$z_{12} = -z_{11}$

$z_{13} = -f_{12}/D$

$z_{14} = (-L\, f_{22} + f_{12})/D$

$z_{22} = z_{11}$

$z_{23} = -\, z_{13}$

$z_{24} = -z_{14}$

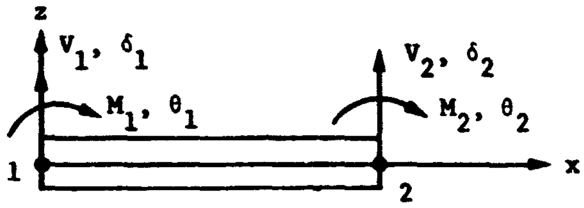
$z_{33} = f_{11}/D$

$z_{34} = (L\, f_{12} - f_{11})/D$

$z_{44} = (L^2\, f_{22} - 2L\, f_{12} + f_{11})/D$

The kernel matrix of Eq [12] is the stiffness matrix that represents the bending and shear stiffness of a beam with unrestrained boundaries.

The elements in the stress transformation matrix are calculated as follows. The following sketch shows the sign convention,



Applying Castigliano's theorem to Eq [12] gives the forces at the beam ends in terms of the displacements at the beam ends, that is,

$$\begin{bmatrix} \partial U/\partial \delta_1 \\ \partial U/\partial \delta_2 \\ \partial U/\partial \theta_1 \\ \partial U/\partial \theta_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ M_1 \\ M_2 \end{bmatrix} = [Z] \begin{bmatrix} \delta_1 \\ \delta_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} .$$

The shear stress is calculated as

$\tau = V/A$

and the bending stress is calculated as

$\sigma = Mc/I$

where c is the distance from the beam's neutral axis to the outer fiber.

Therefore,

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} V_1/A_1 \\ V_2/A_2 \\ M_1c_1/I_1 \\ M_2c_2/I_2 \end{bmatrix} = [TS] \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}$$

where

$$[TS] = \begin{bmatrix} Z \text{ (row 1)}/A_1 \\ Z \text{ (row 2)}/A_2 \\ Z \text{ (row 3) } c_1/I_1 \\ Z \text{ (row 4) } c_2/I_2 \end{bmatrix}$$

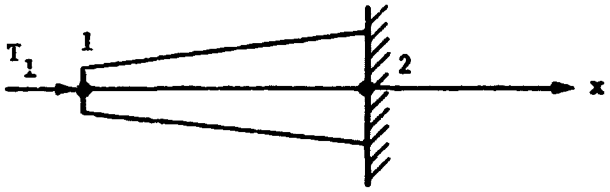
is the stress transformation matrix

Subroutine K1C1 calculates a stiffness matrix and stress transformation matrix for a torsion rod element with unrestrained boundaries. The stiffness matrix is in the local coordinate system of the rod. The elements of the stiffness matrix represent the distributed stiffness properties of the rod. These elements are calculated by assuming constant torque. The stress transformation matrix relates stress at the rod ends in the local coordinate system to rotations in the local coordinate system. The rod may be tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed torsion stiffness of a rod by a stiffness matrix is obtained using a strain energy approach as follows.

Consider a rod that is loaded with a torque T_1 at point 1 and restrained at point 2 as shown in the sketch.



Rod Element

The strain energy is defined by

$$U = \frac{1}{2}\int_{x_1}^{x_2} \frac{T^2(x)}{J(x)\ G(x)}\ dx \qquad [1]$$

where

T is the torque

J is Saint Venant's torsion constant,
$J = \pi R^4/2$ for a solid circular section
$J = 2\pi R^3 t$ for a thin walled circular section

G is the shear modulus of elasticity

x is the local coordinate system and longitudinal axis of the rod. The origin is a point 1, that is $x_1 = 0$; $x_2 = L$ (rod length).

To integrate Eq [1], the torque is assumed constant and equal to the torque at point 1, that is,

$$I(x) = I_1 \qquad [2]$$

The shear modulus of elasticity is also assumed constant, that is,

$$G(x) = G \qquad [3]$$

Saint Venant's torsion constant is assumed to vary linearly, that is,

$$J(x) = J_1 + x\,(J_2 - J_1)/L. \qquad [4]$$

Substituting Eq [2] through [4] into [1] gives the strain energy as

$$U = \frac{1}{2}\,\frac{T_1^2}{G}\int_0^L \frac{1}{J_1 + x(J_2 - J_1)/L}\,dx \qquad [5]$$

$$= \frac{1}{2}\,\frac{T_1^2\,L}{(J_2 - J_1)\,G}\,\ln\,(J_2/J_1) \qquad [6]$$

Application of Castigliano's theorem gives the rotation of point 1 relative to point 2 as

$$\Delta\theta = \frac{\partial U}{\partial T_1} = \frac{T_1\,L}{(J_2 - J_1)\,G}\,\ln\,(J_2/J_1)$$

from which $T_1 = \dfrac{(J_2 - J_1)\,G}{L\,\ln\,(J_2/J_1)}\,\Delta\theta$ [7]

The restraint at point 2 is removed by application of the transformation

$$\Delta\theta = [1 \quad -1] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad [8]$$

where θ_1 and θ_2 are the x-rotations at rod ends 1 and 2, respectively. Substitution of Eq [7] and [8] into [6] gives the strain energy for a rod with unrestrained boundaries as

$$U = \frac{1}{2}\,[\theta_1 \; \theta_2] \begin{bmatrix} z_{1,1} & z_{1,2} \\ (\text{sym}) & z_{2,2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad [9]$$

where

$$z_{1,1} = \frac{(J_2-J_1)\ G}{L\ \ln\ (J_2/J_1)} \qquad [10]$$

$$z_{1,2} = -\ z_{1,1} \qquad [11]$$

$$z_{2,2} = z_{1,1} \qquad [12]$$

The kernel matrix of Eq [9] is the stiffness matrix that represents the torsional stiffness of a rod with unrestrained boundaries.

For a constant cross section, i.e., $J_1 = J_2 = J$, Eq [10] is of indefinite form. For this case, integration of Eq [5] yields

$$U = \frac{1}{2}\,\frac{T_1^2\ L}{JG} \qquad [6a]$$

from which

$$z_{1,1} = \frac{JG}{L} \qquad [10a]$$

as before

$$z_{1,2} = -\ z_{1,1}$$

$$z_{2,2} = z_{1,1}$$

The elements in the stress transformation matrix are easily calculated. The following sketch shows the sign convention.

The rod end torques $(T_1,\ T_2)$ can be expressed in terms of the rod end rotations $(\theta_1,\ \theta_2)$ as

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} z_{1,1} & z_{1,2} \\ \text{(sym)} & z_{2,2} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \qquad [13]$$

This is obtained from Eq [7] or applying Castigliano's theorem to Eq [9]. The maximum stress is in the outermost fiber and is

$$s_1 = T_1\, r_1/J_1$$

and

$$s_2 = T_2\, r_2/J_2.$$

or

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = [T_s] \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

where

$$[T_s] = \begin{bmatrix} z_{1,1}\; r_1/J_1 & z_{1,2}\; r_1/J_1 \\ z_{2,1}\; r_2/J_2 & z_{2,2}\; r_2/J_2 \end{bmatrix}$$

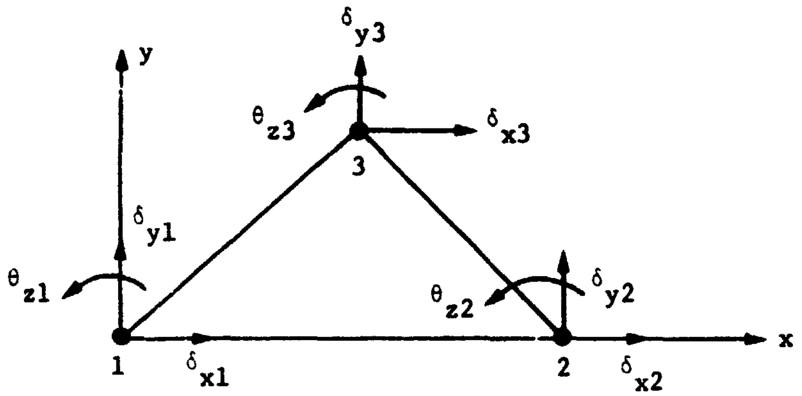
is the stress transformation matrix.

Subroutine K2A1 calculates a stiffness matrix and stress transformation matrix for a membrane triangle plate element with unrestrained boundaries. The stiffness matrix is in the local coordinate system of the triangle plate. The elements of the stiffness matrix represent the distributed stiffness properties of the triangle plate. These elements are calculated by assuming a quadratic displacement (linear strain) field. The stress transformation matrix relates stresses at the triangle vertices in the local coordinate system to deflections in the local coordinate system.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed membrane stiffness of a triangle plate by a stiffness matrix is described in DM 109 *Linear Strain Membrane Triangle Element* by W. A. Benfield and C. S. Bodley.

The triangle is illustrated in the sketch with the degrees of freedom shown



The order of the degrees of freedom is

$$\left[[\delta_x \; \delta_y \; \theta_z]_1 \; [\delta_x \; \delta_y \; \theta_z]_2 \; [\delta_x \; \delta_y \; \theta_z]_3 \right] .$$

The row order of the stress transformation matrix is

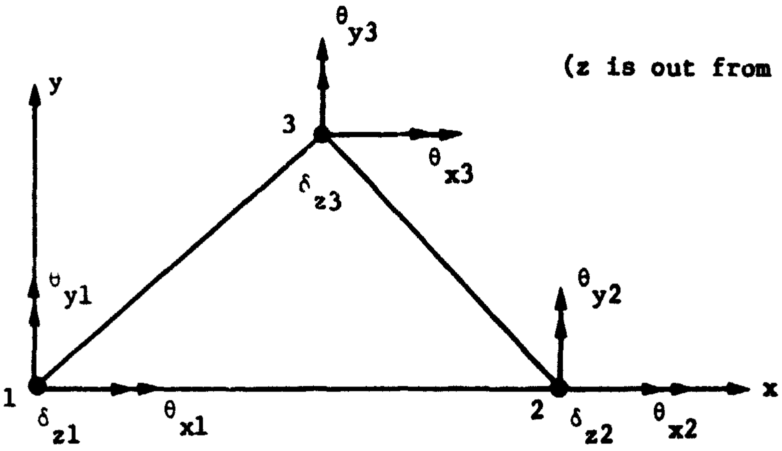
$$\left[[\sigma_x \; \sigma_y \; \tau_{xy}]_1 \; [\sigma_x \; \sigma_y \; \tau_{xy}]_2 \; [\sigma_x \; \sigma_y \; \tau_{xy}]_3 \right]$$

where σ is normal stress and τ is shear stress.

Subroutine K2B1 calculates a stiffness matrix and stress transformation matrix for a bending triangle plate element with unrestrained boundaries. The stiffness matrix is in the local coordinate system of the triangle plate. The elements of the stiffness matrix represent the distributed stiffness properties of the triangle plate. These elements are calculated by assuming a cubic displacement (linear curvature) field. The stress transformation matrix relates stresses at the triangle vertices in the local coordinate system to deflections in the local coordinate system.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed bending stiffness of a triangle plate by a stiffness matrix uses the technique (essentially) described in *Triangular Elements in Plate Bending* by C. P. Bazely, Y. K. Cheung, B. M. Irons, and O. C. Zienkiewicz, AFFDL-TR-66-80, November 1966. The triangle is illustrated in the sketch with the degrees of freedom shown



The order of the degrees of freedom is

$$\left[[\delta_z\ \theta_x\ \theta_y]_1\ [\delta_z\ \theta_x\ \theta_y]_2\ [\delta_z\ \theta_x\ \theta_y]_3\right] .$$

The row order of the stress transformation matrix is

$$\left[[\sigma_x \ \sigma_y \ \tau_{xy}]_1 \quad [\sigma_x \ \sigma_y \ \tau_{xy}]_2 \quad [\sigma_x \ \sigma_y \ \tau_{xy}]_3 \right] \text{ at } z = -\,t/2, \text{ and}$$

$$\left[[\sigma_x \ \sigma_y \ \tau_{xy}]_1 \quad [\sigma_x \ \sigma_y \ \tau_{xy}]_2 \quad [\sigma_x \ \sigma_y \ \tau_{xy}]_3 \right] \text{ at } z = +\,t/2,$$

where

σ is normal stress

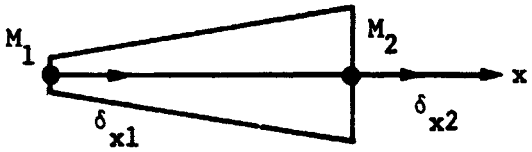
τ is shear stress

t is plate thickness.

Subroutine M1A1 calculates a lumped mass matrix for an axial rod element with unrestrained boundaries. The mass matrix is in the local coordinate system of the rod. The elements of the mass matrix represent the distributed mass properties of the rod. These matrix elements are calculated by *lumping* the rod's total mass to the rod end points using static equivalent forces. The rod may be linearly tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a rod by a mass matrix is obtained by lumping the rod's total mass to the rod end points. This lumping is equivalent to static beaming. The rod is illustrated in the sketch in which the rod cross-sectional area varies linearly, that is, $A(x) = A_1 + x/L\ (A_2-A_1)$.



The total mass of the rod is

$$M = \rho\ L\ (A_1+A_2)/2 \qquad [1]$$

where

ρ is the mass density

A is the cross-sectional area

L is the rod length

x is the local coordinate system and longitudinal axis of the rod with origin at point 1, that is $x_1 = 0$; $x_2 = L$.

The equivalent mass at rod end 2 is calculated by taking the first moment about rod end 1.

$$M_2L = \int_o^L \rho Axdx$$

$$= \int_o^L \rho \, (A_1 + \frac{x}{L} (A_2 - A_1)) \; x \; dx$$

$$= \frac{\rho L^2}{6} (A_1 + 2 \; A_2)$$

from which

$$M_2 = \frac{\rho L}{6} (A_1 + 2 \; A_2) \qquad [2]$$

and $M_1 = M - M_2$

$$= \frac{\rho L}{6} (2A_1 + A_2). \qquad [3]$$

The kinetic energy of the rod element is expressed as

$$T = \frac{1}{2} [\dot{\delta}_{x1}(t) \quad \dot{\delta}_{x2}(t)] \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \dot{\delta}_{x1}(t) \\ \dot{\delta}_{x2}(t) \end{bmatrix} \qquad [4]$$

The kernel matrix of Eq [4] is the mass matrix that represents the distributed mass of the rod.

Subroutine M1A2 calculates a consistent mass matrix for an axial rod element with unrestrained boundaries. The mass matrix is in the local coordinate system of the rod. The elements of the mass matrix represent the distributed mass properties of the rod. These matrix elements are calculated by assuming the displacement between the rod ends to be a *linear* function of the displacement at the rod ends. The rod may be linearly tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a rod by a mass matrix is obtained using a kinetic energy approach as follows.

For small deflections (δ) along the longitudinal axis (x) of the rod shown in the sketch, the kinetic energy is defined by

$$T = \frac{1}{2} \int_{x_1}^{x_2} \rho A(x) \; \dot{\delta}^2 (x,t) \, dx \qquad [1]$$

where

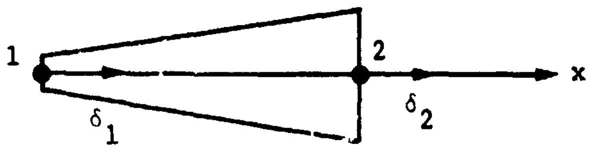
ρ is the mass density

A is the cross-sectional area

$\dot{\delta}$ is the time rate of change of displacement along the rod x-axis, referred to as longitudinal velocity in the paper

t is time

x is the local coordinate system and longitudinal axis of the rod with origin at point 1, that is, $x_1 = 0$; $x_2 = L$ (rod length).



Rod Element

To integrate Eq [1] a *linear* displacement function will be assumed between points 1 and 2 in terms of the displacements of points 1 and 2, that is,

$$\delta(x) = \delta_1 + \frac{x}{L}\,(\delta_2 - \delta_1).$$

Similarly, the longitudinal velocity is given by

$$\dot{\delta}(x,t) = \dot{\delta}_1(t) + \frac{x}{L}\,(\dot{\delta}_2(t) - \dot{\delta}_1(t))$$

$$= \frac{1}{L}\,[(L-x) \quad x] \begin{bmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{bmatrix} . \qquad [2]$$

The cross-sectional area of the rod is assumed to vary linearly, that is,

$$A(x) = A_1 + \frac{x}{L}\,(A_2 - A_1). \qquad [3]$$

Substituting Eq [2] and [3] into [1] gives the kinetic energy as

$$T = \frac{1}{2}\,[\dot{\delta}_1(t) \quad \dot{\delta}_2(t)]\,\frac{\rho}{L^2}\left(\int_o^L \begin{bmatrix} L-x \\ x \end{bmatrix} \left(A_1 + \frac{x}{L}\,(A_2 - A_1)\right) [(L-x) \quad x]\,dx\right) \begin{bmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{bmatrix} = \frac{1}{2}\,[\dot{\delta}_1(t) \quad \dot{\delta}_2(t)]\,\frac{\rho L}{12} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix} \begin{bmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \end{bmatrix} \qquad [4]$$

The kernel matrix of Eq [4] is the mass matrix.

The total mass properties of the rod may be calculated from the following triple matrix product.

$$\begin{bmatrix} M & P^o \\ P^o & I^o \end{bmatrix} = \frac{\rho L}{12} \begin{bmatrix} 1 & 1 \\ 0 & L \end{bmatrix} \begin{bmatrix} 3A_1 + A_2 & A_1 + A_2 \\ A_1 + A_2 & A_1 + 3A_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \qquad [5]$$

where

M is the mass of the rod

P^o is the first moment about $x_1 = 0$

I^o is the moment of inertia about $x_1 = 0$.

Expanding the triple matrix product gives

$$M = \rho \ (A_1+A_2) \ L/2 \qquad [5a]$$

$$P^o = \rho \ (A_1+A_2) \ L^2/6 \qquad [5b]$$

$$I^o = \rho \ (A_1+ 3A_2) \ L^3/12 \qquad [5c]$$

The center of gravity is calculated from

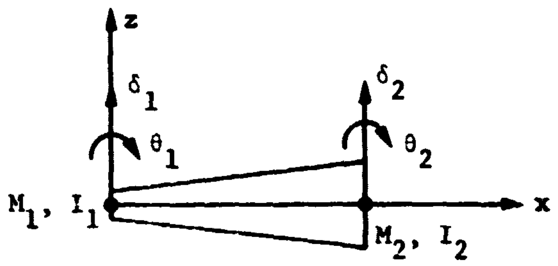
$$x_{cg} = P^o/M$$

$$= \frac{(A_1+ 2A_2) \ L}{(A_1+A_2) \ 3} \ . \qquad [5d]$$

Subroutine M1B1 calculates a lumped mass matrix for a bending beam element with unrestrained boundaries. The mass matrix is in the local coordinate system of the beam. The elements of the mass matrix represent the distributed mass properties of the beam. These matrix elements are calculated by lumping the beam's total mass to the beam end points using static equivalent forces. The beam may be linearly tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a beam by a mass matrix is obtained by lumping the beam's total mass to the beam end points. This lumping is equivalent to static beaming. The beam is illustrated in the sketch in which the rod cross-sectional area varies linearly, that is, $A(x) = A_1 + x/L\ (A_2-A_1)$.



(local y is into paper)

The total mass of the beam is

$$M = \rho\ L(A_1+A_2)/2 \qquad [1]$$

where

ρ is the mass density

A is the cross-sectional area

L is the beam length

x is the local coordinate system and longitudinal axis of the beam with origin at point 1, that is, $x_1 = 0$; $x_2 = L$.

The equivalent mass at beam end 2 is calculated by taking the first moment about beam end 1.

$$M_2L = \int_0^L \rho \ Ax \ dx$$

$$= \int_0^L \rho \ (A_1 + \frac{x}{L} (A_2-A_1)) \ x \ dx$$

$$= \frac{\rho L^2}{6} \ (A_1 + 2A_2)$$

$$M_2 = \rho L \ (A_1 + 2A_2)/6 \qquad [2]$$

From

$$M_1 + M_2 = M$$

$$M_1 = \rho L \ (2A_1+A_2)/6 \qquad [3]$$

An attempt at calculating the inertias was made by taking the second moment about beam end 1.

$$I_1 + I_2 + M_2 \ L^2 = \int_0^L \rho A \ x^2 \ dx$$

$$= ML^2/3$$

$$I_1 + I_2 = (M_1-2M_2) \ L^2/3$$

For a uniform beam, $M_1 = M_2 = M/2$

$$I_1 + I_2 = - ML^2/6$$

which is impossible. This says that lumping can never give the correct inertia values. Therefore, arbitrarily assume

$$I_1 = \rho A_1 \ L^3/24 \qquad [4]$$

and

$$I_2 = \rho A_2 \, L^3/24 \qquad [5]$$

The kinetic energy of the beam element is expressed as

$$T = \frac{1}{2} \text{ [same as column] } \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & I_1 & \\ & & & I_2 \end{bmatrix} \begin{bmatrix} \dot{\delta}_1 \\ \dot{\delta}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \qquad [6]$$

The kernel matrix of [6] is the mass matrix that represents the distributed mass of the beam.

Subroutine M1B2 calculates a consistent mass matrix for a bending beam element with unrestrained boundaries. The mass matrix is in the local coordinate system of the beam. The elements of the mass matrix represent the distributed mass properties of the beam. The matrix elements are calculated by assuming the displacement between the rod ends to be a *cubic* function of the displacement at the beam ends. The beam may be linearly tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a beam by a mass matrix is obtained using a kinetic energy approach as follows.

For small deflections (δ) normal to the longitudinal axis (x) of the beam shown in the sketch, the kinetic energy is defined by

$$T = \frac{1}{2}\int_{x_1}^{x_2} \rho\ A(x)\ \dot{\delta}^2\ (x,t)\ dx \qquad [1]$$

where

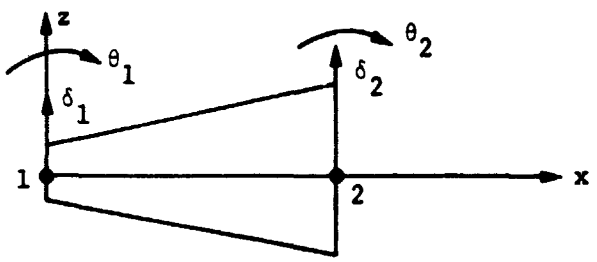
ρ is the mass density

A is the cross-sectional area

$\dot{\delta}$ is the time rate of change of displacement normal to the body x-axis, referred to as lateral velocity in this paper

t is time

x is the local coordinate system and longitudinal axis of the beam with origin at point 1, that is, $x_1 = 0$; $x_2 = L$ (beam length).



To integrate Eq [1] a cubic displacement function is assumed between points 1 and 2 in terms of the displacements of points 1 and 2, that is,

$$\begin{aligned}\delta(x) &= Ax^3 + Bx^2 + Cx + D \\ &= [x^3\ x^2\ x\ 1] \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}\end{aligned} \qquad [2]$$

The angular displacement is obtained as the geometric derivative of the lateral displacement, that is

$$\begin{aligned}\theta(x) &= - \frac{\partial\ \delta(x)}{\partial x} \\ &= [-3x^2\ -2x\ -1\ 0] \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}\end{aligned} \qquad [3]$$

The coefficients A, B, C, D are determined from Eq [2] and [3], using the displacements at the beam ends.

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ L^3 & L^2 & L & 1 \\ 0 & 0 & -1 & 0 \\ -3L^2 & -2L & -1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

From which

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = [\psi] \begin{bmatrix} \delta_1 \\ \delta_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} \qquad [4a]$$

where

$$[\psi] = \begin{bmatrix} 2/L^3 & -2/L^3 & -1/L^2 & -1/L^2 \\ -3/L^2 & 3/L^2 & 2/L & 1/L \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \qquad [4b]$$

Using Eq [4a] in [2] and taking the time derivative gives the lateral velocity as

$$\dot{\delta}(x,t) = [x^3 \; x^2 \; x \; 1] \; [\psi] \begin{bmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \\ \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix} \qquad [5]$$

The cross-sectional area of the beam is assumed to vary linearly, that is,

$$A(x) = A_1 + \frac{x}{L} (A_2 - A_1). \qquad [6]$$

Substituting Eq [5] and [6] into [1] gives the kinetic energy as

$$T = \frac{1}{2} [\text{same as column}]^T \cdot [\psi]^T \left(\int_0^L \begin{bmatrix} x^6 & x^5 & x^4 & x^3 \\ x^5 & x^4 & x^3 & x^2 \\ x^4 & x^3 & x^2 & x \\ x^3 & x^2 & x & 1 \end{bmatrix} A_1 + \frac{x}{L} (A_2 - A_1) \; dx \right) [\psi] \begin{bmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \\ \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix}$$

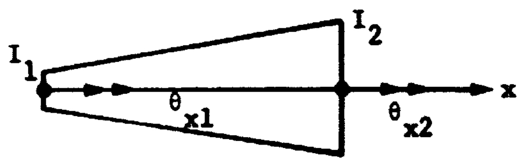
$$= \frac{1}{2} [\text{same as column}]^T \frac{\rho L}{840} \begin{bmatrix} 240A_1+72A_2 & 54A_1+54A_2 & -(30A_1+14A_2)L & (14A_1+12A_2)L \\ & 72A_1+240A_2 & -(12A_1+14A_2)L & (14A_1+30A_2)L \\ (\text{sym}) & & (5A_1+3A_2)L^2 & -(3A_1+3A_2)L^2 \\ & & & (3A_1+5A_2)L^2 \end{bmatrix} \begin{bmatrix} \dot{\delta}_1(t) \\ \dot{\delta}_2(t) \\ \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix} \qquad [7]$$

The kernel matrix of Eq [7] is the mass matrix

Subroutine M1C1 calculates a lumped mass matrix for a torsion rod element with unrestrained boundaries. The mass matrix is in the local coordinate system of the rod. The elements of the mass matrix represent the distributed inertia properties of the rod. These matrix elements are calculated by *lumping* the rod's total inertia to the rod end points using static equivalent forces. The rod may be linearly tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed inertia of a rod by a mass matrix is obtained by lumping the rods total inertia to the rod end points. This lumping is equivalent to static beaming. The rod is illustrated in the sketch in which the rod cross-sectional polar area moment of inertia varies linearly, that is, $P(x) = P_1 + x/L\ (P_2-P_1)$.



The total inertia of the rod is

$$I = \rho L\ (P_1+P_2)/2 \qquad [1]$$

where

ρ is the mass density

P is the cross-sectional polar area moment of inertia

L is the rod length

x is the local coordinate system and longitudinal axis of the rod with origin at point 1, that is, $x_1 = 0$; $x_2 = L$.

The equivalent inertia at rod end 2 is calculated by taking the first moment about rod end 1.

$$I_2 \, L = \int_o^L \rho P x \, dx$$

$$= \int_o^L \rho \; (P_1 + \frac{x}{L} (P_2 - P_1)) \; x \, dx$$

$$= \frac{\rho \, L^2}{6} \; (P_1 + 2P_2)$$

from which

$$I_2 = \frac{\rho L}{6} (P_1 + P_2) \qquad [2]$$

and

$$I_1 = I - I_2$$

$$= \frac{\rho L}{6} (2P_1 + P_2). \qquad [3]$$

The kinetic energy of the rod element is expressed as

$$T = \frac{1}{2} [\dot{\theta}_{x1}(t) \;\; \dot{\theta}_{x2}(t)] \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_{x1}(t) \\ \dot{\theta}_{x2}(t) \end{bmatrix} \qquad [4]$$

The kernel matrix of Eq [4] is the mass matrix that represents the distributed inertia of the rod.

Subroutine M1C2 calculates a consistent mass matrix for a torsion rod element with unrestrained boundaries. The mass matrix is in the local coordinate system at the rod. The elements of the mass matrix represent the distributed inertia properties of the rod. These matrix elements are calculated by assuming the displacement between the rod ends to be a *linear* function of the displacement at the rod ends. The rod may be linearly tapered or uniform.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed inertia of a rod by a mass matrix is obtained using a kinetic energy approach as follows.

For small rotations (θ) along the longitudinal axis (x) of the rod shown in the sketch, the kinetic energy is defined by

$$T = \frac{1}{2}\int_{x_1}^{x_2} \rho\, P(x)\, \dot{\theta}^2 (x,t)\, dx \qquad [1]$$

where

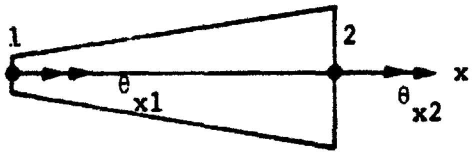
ρ is the mass density

P is the cross-sectional polar area moment of inertia

$\dot{\theta}$ is the time rate of change of rotation along the rod x-axis, referred to as rotational velocity in this paper

t is time

x is the local coordinate system and longitudinal axis of the rod with origin at point 1, that is, $x_1 = 0$; $x_2 = L$ (rod length).



Rod Element

To integrate Eq [1] a *linear* displacement function will be assumed between points 1 and 2 in terms of the rotations of points 1 and 2, that is,

$\theta(x) = \theta_1 + \frac{x}{L}(\theta_2 - \theta_1)$.

Similarly the rotational velocity is given by

$$\dot{\theta}(x,t) = \dot{\theta}_1(t) + \frac{x}{L}\ (\dot{\theta}_2(t) - \dot{\theta}_1(t) = \frac{1}{L}\,[(L-x)\quad x]\begin{bmatrix}\dot{\theta}_1(t)\\ \dot{\theta}_2(t)\end{bmatrix}. \qquad [2]$$

The cross-sectional polar area moment of inertia of theorem is assumed to vary linearly, that is,

$$P(x) = P_1 + \frac{x}{L}\,(P_2-P_1). \qquad [3]$$

Substituting Eq [2] and [3] into [1] gives the kinetic energy as

$$T = \frac{1}{2}\,[\dot{\theta}_1(t)\ \ \dot{\theta}_2(t)]\,\frac{\rho}{L^2}\left(\int_0^L \begin{bmatrix}L-x\\ x\end{bmatrix}\left(P_1 + \frac{x}{L}\ \ P_2-P_1\right)[(L-x)\ x]\ dx\right)\begin{bmatrix}\dot{\theta}_1(t)\\ \dot{\theta}_2(t)\end{bmatrix}$$

$$= \frac{1}{2}\,[\dot{\theta}_1(t)\ \ \dot{\theta}_2(t)]\,\frac{\rho L}{12}\begin{bmatrix}3P_1+P_2 & P_1+P_2\\ (\mathrm{sym}) & P_1+3P_2\end{bmatrix}\begin{bmatrix}\dot{\theta}_1(t)\\ \dot{\theta}_2(t)\end{bmatrix} \qquad [4]$$

The kernel matrix of Eq [4] is the mass matrix.

The total inertia of the rod can be calculated by assuming a rigid body mode of

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}, = \begin{bmatrix}1\\ 1\end{bmatrix}\dot{\theta} \qquad [5]$$

Substitution of [5] into [4] gives $T = \frac{1}{2}\,I\,\dot{\theta}^2$,

where

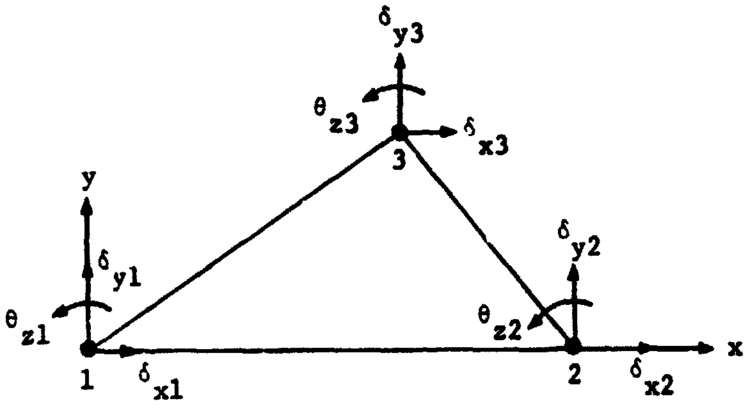
$$I = [1\quad 1]\ \frac{\rho L}{12}\begin{bmatrix}3P_1+P_2 & P_1+P_2\\ P_1+P_2 & P_1+3P_2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$= \rho\ (P_1+P_2)/L2. \qquad [6]$$

Subroutine M2A1 calculates a lumped mass matrix for a membrane triangle plate element with unrestrained boundaries. The mass matrix is in the local coordinate system of the triangle plate. The elements of the mass matrix represent the distributed mass properties of the triangle.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a membrane triangle plate by a mass matrix is obtained by lumping the triangle's total mass to the triangle vertices. The triangle is illustrated in the sketch with the degrees of freedom shown.



The total mass of the triangle is

$$M = \rho t\, x_2\, y_3/2 \qquad [1]$$

where

ρ is the mass density

t is the plate thickness.

The mass at each vertex for a translation degree of freedom is M/3. Because any inertia at a vertex will always be "heavy", arbitrarily assume this value to be M/3 also.

The kinetic energy of the membrane triangle plate element is expressed as

$$T = \frac{1}{2}\,[\text{same as column}]^T \begin{bmatrix} M/3 &&&&&&&& \\ & M/3 &&&&&&& \\ && M/3 &&&&&& \\ &&& M/3 &&&&& \\ &&&& M/3 &&&& \\ &&&&& M/3 &&& \\ &&&&&& M/3 && \\ &&&&&&& M/3 & \\ &&&&&&&& M/3 \end{bmatrix} \begin{bmatrix} \begin{Bmatrix} \dot{\delta}_x \\ \dot{\delta}_y \\ \dot{\theta}_z \end{Bmatrix}_1 \\ \begin{Bmatrix} \dot{\delta}_x \\ \dot{\delta}_y \\ \dot{\theta}_z \end{Bmatrix}_2 \\ \begin{Bmatrix} \dot{\delta}_x \\ \dot{\delta}_y \\ \dot{\theta}_z \end{Bmatrix}_3 \end{bmatrix} \qquad [2]$$

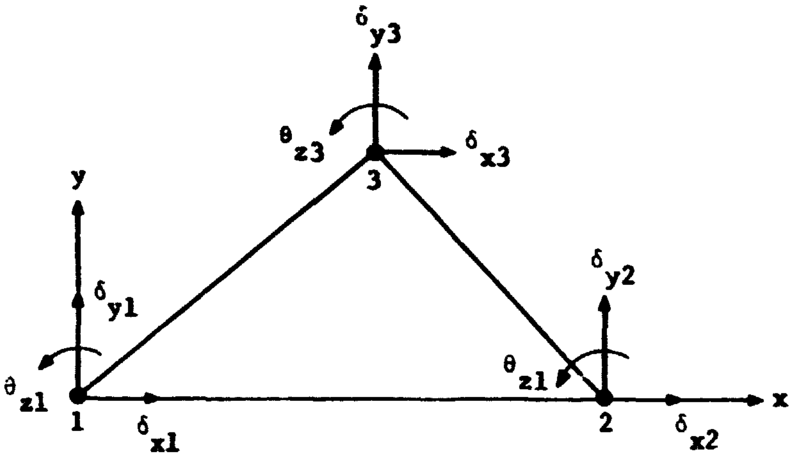
The kernel matrix of Eq [2] is the mass matrix.

Subroutine M2A2 calculates a consistent mass matrix for a membrane triangle plate element with unrestrained boundaries. The mass matrix is in the local coordinate system of the triangle plate. The elements of the mass matrix represent the distributed mass properties of the triangle. These matrix elements are calculated by assuming a quadratic displacement field.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a membrane triangle plate by a mass matrix is described in D.M. 169 *Linear Strain Membrane Triangle Element* by W. A. Benfield and C. S. Bodley.

The triangle is illustrated in the sketch with the degrees of freedom shown.



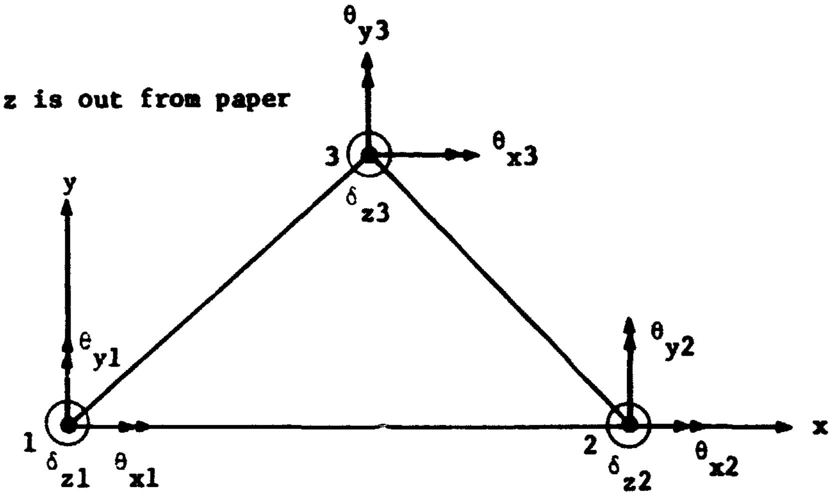
The order of the degrees of freedom is

$$\left[[\delta_x\ \delta_y\ \theta_z]_1\ [\delta_x\ \delta_y\ \theta_z]_2\ [\delta_x\ \delta_y\ \theta_z]_3 \right] .$$

Subroutine M2B1 calculates a lumped mass matrix for a bending triangle plate element with unrestrained boundaries. The mass matrix is in the local coordinate system of the triangle plate. The elements of the mass matrix represent the distributed mass properties of the triangle.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a bending triangle plate by a mass matrix is obtained by lumping the triangle's total mass to the triangle vertices. The triangle is illustrated in the sketch with the degrees of freedom shown.



The total mass of the triangle is

$$M = \rho t\, x_2 y_3 / 2 \qquad [1]$$

where

ρ is the mass density, and

t is the plate thickness.

The mass at each vertex for a translation degree of freedom is M/3. Because any inertia at a vertex will always be "heavy", arbitrarily assume this value to be M/3 also.

The kinetic energy of the bending triangle plate element is expressed as

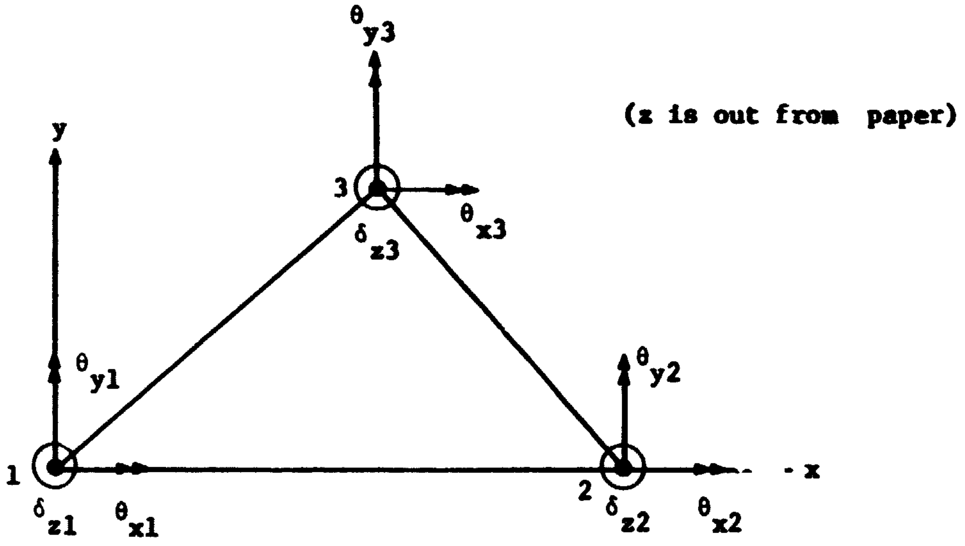
$$T = \frac{1}{2}\,[\text{same as column}]^T \begin{bmatrix} M/3 &&&&&&&& \\ & M/3 &&&&&&& \\ && M/3 &&&&&& \\ &&& M/3 &&&&& \\ &&&& M/3 &&&& \\ &&&&& M/3 &&& \\ &&&&&& M/3 && \\ &&&&&&& M/3 & \\ &&&&&&&& M/3 \end{bmatrix} \begin{bmatrix} \begin{Bmatrix} \dot{\delta}_z \\ \dot{\theta}_x \\ \dot{\theta}_y \end{Bmatrix}_1 \\ \begin{Bmatrix} \dot{\delta}_z \\ \dot{\theta}_x \\ \dot{\theta}_y \end{Bmatrix}_2 \\ \begin{Bmatrix} \dot{\delta}_z \\ \dot{\theta}_x \\ \dot{\theta}_y \end{Bmatrix}_3 \end{bmatrix} \quad [2]$$

The kernel matrix of Eq [2] is the mass matrix.

Subroutine M2B2 calculates a consistent mass matrix for a bending triangle plate element with unrestrained boundaries. The mass matrix is in the local coordinate system of the triangle plate. The elements of the mass matrix represent the distributed mass properties of the triangle. These matrix elements are calculated by assuming a cubic displacement field.

DESCRIPTION OF TECHNIQUE

The replacement of the distributed mass of a bending triangle plate by a mass matrix uses the technique (essentially) described in *Trianglular Elements in Plate Bending* by C. P. Bazely, Y. K. Cheung, B. M. Irons, and O. C. Zienkiewicz; AFFDL-TR-66-80, November 1966. The triangle is illustrated in the sketch with the degrees of freedom shown.



The order of the degrees of freedom is

$$\left[[\delta_z \ \theta_x \ \theta_y]_1 \ [\delta_z \ \theta_x \ \theta_y]_2 \ [\delta_z \ \theta_x \ \theta_y]_3 \right] .$$

Subroutine MAS1A calculates a finite element mass matrix for an axial rod element with unrestrained boundaries.

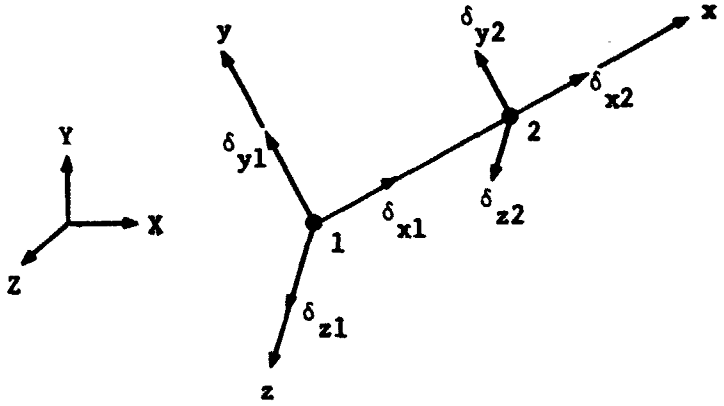
The mass matrix, size 6x6, is in the global coordinate directions. The global coordinate order for each element is (U, V, W) joint 1, then joint 2 where U, V, W are translations. If the Euler angles are zero at a joint, $U = \delta_X$, $V = \delta_Y$, $W = \delta_Z$. The mass matrix may be either lumped or consistent.

This mass matrix is computed by first calculating a mass matrix in the local coordinate system for either a lumped mass matrix or a consistent mass matrix. Euler angles are then used to transform the mass matrix from the local coordinate system to the global coordinate directions. Direction cosines are not needed.

DESCRIPTION OF TECHNIQUE

The calculation of the mass matrix in the global coordinate directions is accomplish as follows. First a mass matrix is calculated in the local coordinate system and can be given as $[m_L] = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ for either the lumped mass matrix (reference subroutine M1A1) or the consistent mass matrix (reference subroutine M1A2). The off-diagonal terms are zero for the lumped mass matrix.

A sketch of the rod is given for reference as



The kinetic energy using local coordinates is

$$T = \frac{1}{2}\ [\text{same as column}]^T \left[\begin{array}{c|c} m_{11}\ [I] & m_{12}\ [I] \\ \hline m_{21}\ [I] & m_{22}\ [I] \end{array}\right] \left[\begin{array}{c} \dot{\delta}_{x1} \\ \dot{\delta}_{y1} \\ \dot{\delta}_{z1} \\ \dot{\delta}_{x2} \\ \dot{\delta}_{y2} \\ \dot{\delta}_{z2} \end{array}\right] \qquad [1]$$

where [I] is a unity matrix, size 3x3.

The deflections in the local system are related to the deflections in the global coordinate directions by

$$\left[\begin{array}{c} \delta_{x1} \\ \delta_{y1} \\ \delta_{z1} \\ \hline \delta_{x2} \\ \delta_{y2} \\ \delta_{z2} \end{array}\right] = \left[\begin{array}{c|c} [\gamma]\ [E_1] & 0 \\ \hline 0 & [\gamma]\ [E_2] \end{array}\right] \{h_G\} \qquad [2]$$

where

$[\gamma]$ is a direction cosine matrix, size 3x3,

$[E_i]$ is an Euler angle transformation matrix (reference subroutine EULER) at joint i, size 3x3, and

$\{h_G\} = [U_1\ V_1\ W_1\ U_2\ V_2\ W_2]$. U, V, W are translations.

Substituting Eq. [2] into [1] gives the kinetic energy using global coordinates as

$$T = \frac{1}{2}\ \{\dot{h}_G\}^T \left[\begin{array}{c|c} m_{11}\ [E_1]^T\ [\gamma]^T\ [\gamma]\ [E_1] & m_{12}\ [E_1]^T\ [\gamma]^T\ [\gamma]\ [E_2] \\ \hline m_{21}\ [E_2]^T\ [\gamma]^T\ [\gamma]\ [E_1] & m_{22}\ [E_2]^T\ [\gamma]^T\ [\gamma]\ [E_2] \end{array}\right] \{h_G\}$$

[3]

The kinetic energy using local coordinates is

$$T = \frac{1}{2}\,[\text{same as column}]^T \left[\begin{array}{c:c} m_{11}\ [I] & m_{12}\ [I] \\ \hdashline m_{21}\ [I] & m_{22}\ [I] \end{array}\right] \begin{bmatrix} \dot{\delta}_{x1} \\ \dot{\delta}_{y1} \\ \dot{\delta}_{z1} \\ \dot{\delta}_{x2} \\ \dot{\delta}_{y2} \\ \dot{\delta}_{z2} \end{bmatrix} \qquad [1]$$

where [I] is a unity matrix, size 3x3.

The deflections in the local system are related to the deflections in the global coordinate directions by

$$\left[\begin{array}{c} \delta_{x1} \\ \delta_{y1} \\ \delta_{z1} \\ \hdashline \delta_{x2} \\ \delta_{y2} \\ \delta_{z2} \end{array}\right] = \left[\begin{array}{c:c} [\gamma]\ [E_1] & 0 \\ \hdashline 0 & [\gamma]\ [E_2] \end{array}\right] \{h_G\} \qquad [2]$$

where

$[\gamma]$ is a direction cosine matrix, size 3x3,

$[E_i]$ is an Euler angle transformation matrix (reference subroutine EULER) at joint i, size 3x3, and

$\{h_G\} = [U_1\ V_1\ W_1\ U_2\ V_2\ W_2]$. U, V, W are translations.

Substituting Eq. [2] into [1] gives the kinetic energy using global coordinates as

$$T = \frac{1}{2}\,\{\dot{h}_G\}^T \left[\begin{array}{c:c} m_{11}\ [E_1]^T\ [\gamma]^T\ [\gamma]\ [E_1] & m_{12}\ [E_1]^T\ [\gamma]^T\ [\gamma]\ [E_2] \\ \hdashline m_{21}\ [E_2]^T\ [\gamma]^T\ [\gamma]\ [E_1] & m_{22}\ [E_2]^T\ [\gamma]^T\ [\gamma]\ [E_2] \end{array}\right] \{h_G\}$$

[3]

$$= \frac{1}{2} \{\dot{h}_G\}^T \left[\begin{array}{c:c} m_{11} \; [I] & m_{12} \; [E_1]^T \; [E_2] \\ \hdashline m_{21} \; [E_2]^T \; [E_1] & m_{22} \; [I] \end{array}\right] \{\dot{h}_G\} \qquad [4]$$

because $[\gamma] \; [\gamma]^T = [I]$ and $[E_i] \; [E_i]^T = [I]$ since $[\gamma]$ and $[E_i]$ are orthonormal.

For lumped mass, $m_{12}=m_{21}=0$. The kernel matrix in Eq [4] is the desired mass matrix in the global coordinate directions. If the Euler angles are zero at both joints, $[E_i] = [I]$.

Subroutine MAS1B calculates a finite element mass matrix for a combined axial-torsion-bending bar element with unrestrained boundaries.

The mass matrix, size 12x12, is in the global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1, then joint 2 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

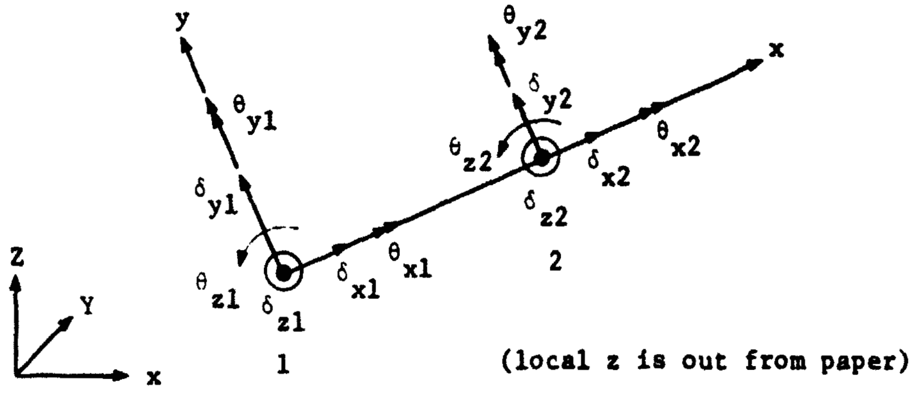
This mass matrix is computed by first calculating a mass matrix in the local coordinate system for either a lumped mass matrix or a consistent mass matrix. A direction cosine matrix is then used to transform the mass matrix from the local coordinate system to the global coordinate directions.

DESCRIPTION OF TECHNIQUE

The calculation of the mass matrix in the global coordinate directions is accomplished as follows. First a mass matrix is calculated in the local coordinate system for either lumped mass or consistent mass using uncoupled axial, torsion, and bending subroutines listed.

	lumped	consistent
axial	M1A1	M1A2
bending	M1B1	M1B2
torsion	M1C1	M1C2

A sketch of the bar is given for reference as



The kinetic energy using local coordinates is

$$T = \frac{1}{2} \{\dot{h}_L\}^T [m_L] \{\dot{h}_L\} \qquad [1]$$

where

$$\{h_L\}^T = [\delta_{x1}\ \delta_{x2}\ \vert\ \theta_{x1}\ \theta_{x2}\ \vert\ \delta_{y1}\ \delta_{y2}\ \theta_{z1}\ \theta_{z2}\ \vert\ \delta_{z1}\ \delta_{z2}\ \theta_{y1}\ \theta_{y2}]$$

and

$$[m_L] = \begin{bmatrix} a_{11} & a_{12} & & & & & & & & & & \\ a_{21} & a_{22} & & & & & & & & & & \\ & & t_{11} & t_{12} & & & & & & & & \\ & & t_{21} & t_{22} & & & & & & & & \\ & & & & b_{11} & b_{12} & -b_{13} & -b_{14} & & & & \\ & & & & b_{21} & b_{22} & -b_{23} & -b_{24} & & & & \\ & & & & -b_{31} & -b_{32} & b_{33} & b_{34} & & & & \\ & & & & -b_{41} & -b_{42} & b_{43} & b_{44} & & & & \\ & & & & & & & & b_{11} & b_{12} & b_{13} & b_{14} \\ & & & & & & & & b_{21} & b_{22} & b_{23} & b_{24} \\ & & & & & & & & b_{31} & b_{32} & b_{33} & b_{34} \\ & & & & & & & & b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

a_{ij}, t_{ij}, b_{ij} refer to terms in the uncoupled axial, torsion, bending mass matrices. For lumped mass, the off-diagonal terms are zero.

The above deflections in the local coordinate directions are related to deflections in the global coordinate directions by

$$\{h_L\} = [\gamma]\ \{h_G\} \qquad [2]$$

where $[\gamma]$ is a direction cosine matrix (reference subroutine DCOS1B) including Euler angles, size 12x12, and $\{h_G\}^T = [U_1 V_1 W_1 P_1 Q_1 R_1\ U_2 V_2 W_2 P_2 Q_2 R_2]$. U, V, W are translations and P, Q, R are rotations.

Substituting Eq [2] into [1] gives the kinetic energy using global coordinates as

$$T = \frac{1}{2} \{\dot{h}_G\}^T [m_G] \{\dot{h}_G\} \qquad [3]$$

where $[m_G] = [\gamma]^T [m_L] [\gamma]$ is the desired mass matrix in the global coordinate directions.

Even though the local lumped mass matrix has only diagonal terms, the triple matrix product using direction cosines is needed because $t_{11} \neq b_{33}$ and $t_{22} \neq b_{44}$.

Subroutine MAS2 calculates a finite element mass matrix for a combined membrane-bending triangle plate element with unrestrained boundaries.

The mass matrix, size 18x18, is in the global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1, then joints 2, 3 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then U=δ_X, V=δ_Y, W=δ_X, P=θ_X, Q=θ_Y, R=θ_Z.

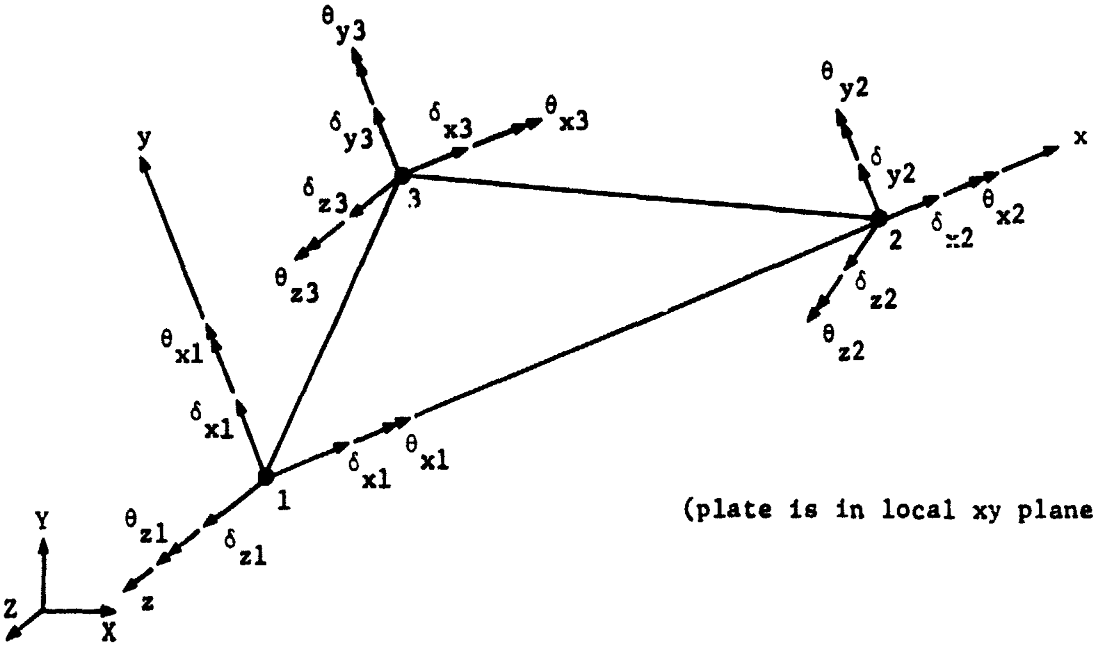
This mass matrix is computed by first calculating a mass matrix in the local coordinate system for either a lumped mass matrix or a consistent mass matrix. A direction cosine matrix is then used to transform the mass matrix from the local coordinate system to the global coordinate directions.

DESCRIPTION OF TECHNIQUE

The calculation of the mass matrix in the global coordinate directions is accomplished as follows. First a mass matrix is calculated in the local coordinate system for either lumped mass or consistent mass using uncoupled membrane and bending subroutines listed.

	lumped	consistent
membrane	M2A1	M2A2
bending	M2B1	M2B2

A sketch of the triangle plate is given for reference as



(plate is in local xy plane)

The kinetic energy using local coordinates is

$$T = \frac{1}{2} \{\dot{h}_L\}^T \; [m_L] \; \{\dot{h}_L\} \qquad [1]$$

where

$$\begin{aligned} \{h_L\}^T = [\delta_{x1} \; \delta_{y1} \; \theta_{z1} \quad \delta_{x2} \; \delta_{y2} \; \theta_{z2} \quad \delta_{x3} \; \delta_{y3} \; \theta_{z3} \; \vdots \\ \delta_{z1} \; \theta_{x1} \; \theta_{y1} \quad \delta_{z2} \; \theta_{x2} \; \theta_{y2} \quad \delta_{z3} \; \theta_{x3} \; \theta_{y3}] \end{aligned}$$

and

$$[m_L] = \left[\begin{array}{c:c} [m_L]_{mem} & 0 \\ \hdashline 0 & [m_L]_{ben} \end{array} \right]$$

The above deflections in the local coordinate directions are related to deflections in the global coordinate directions by

$$\{h_L\} = [\gamma] \; \{h_G\} \qquad [2]$$

where $[\gamma]$ is a direction cosine matrix (reference subroutine DCOS2) including Euler angles, size 18x18, and $\{h_G\}^T =$ $[U_1 \; V_1 \; W_1 \; P_1 \; Q_1 \; R_1 \; U_2 \; V_2 \; W_2 \; P_2 \; Q_2 \; R_2]$. U, V, W are translations and P, Q, R are rotations.

Substituting Eq [2] into [1] gives the kinetic energy using global coordinates as

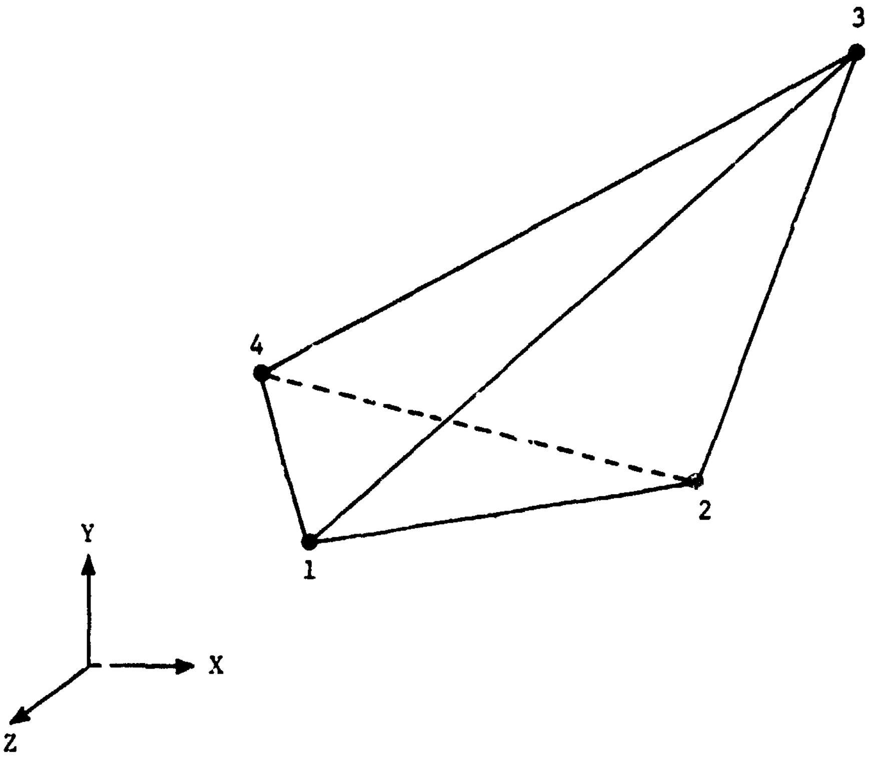
$$T = \frac{1}{2} \{\dot{h}_G\}^T \; [m_G] \; \{\dot{h}_G\} \qquad [3]$$

where $[m_G] = [\gamma]^T \; [m_L] \; [\gamma]$ is the desired mass matrix in the global coordinate directions.

Because the local lumped mass matrix is diagonal and because translation terms are equal for membrane and bending, and rotation terms are equal for membrane and bending, the triple matrix product using direction cosines is not needed. A simple reordering of diagonal terms is sufficient, Subroutine MAS3 calculates a finite element mass matrix for a combined membrane-bending quadrilateral plate element with unrestrained boundaries.

The mass matrix, size 24x24, is in the global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1, then joints 2, 3, 4 where U, V, W, are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

This mass matrix is calculated by taking the average overlap of four triangles shown in the sketch. Either a lumped mass matrix or a consistent mass matrix is calculated. Subroutine MAS2 is used for the calculation of the mass matrix for the triangular plates.



QUAD

Subroutine QUAD calculates (on option) finite element: (1) mass matrices; (2) stiffness matrices (same as global load transformation matrices), and (3) vectors to locate the DOF (degrees of freedom) of the matrices in the global DOF, for combined membrane-bending quadrilateral plate elements. The above matrices and vectors are written on disk units and constitute the output from this subroutine. All matrices are in dense programming logic.

Each mass and stiffness matrix, size 24x24, is in the global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1, then joints 2, 3, and 4 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

Each global load transformation matrix, size 24x24, relates loads at quadrilateral vertices in the global coordinate directions to deflections in the global coordinate directions. The row order in this matrix is (P_U, P_V, P_W, M_P, M_Q, M_R) joint 1, then joints 2, 3 and 4 where P is force and M is moment.

Each location vector (IVEC) locates the DOF of each finite element in the global DOF. For example, IVEC(6)=834 places element DOF 6 into global DOF 834. IVEC(3)=0 omits element DOF 3 from global DOF. This constrains element DOF 3 to zero motion.

The above matrices are calculated by using joint data and element data. The joint data is obtained from three matrices input to this subroutine: (1) joint global X, Y, Z locations; (2) joint global DOF numbers; and (3) joint Euler angles.

The element data, read in this subroutine, is: (1) options for mass, stiffness; (2) element material properties; and (3) element joint numbers.

Each mass matrix is calculated by transfer to subroutine MAS3. Each stiffness matrix is calculated by transfer to subroutine STF3.

Subroutine STF1A calculates a finite element; (1) stiffness matrix (same as global load transformation matrix); (2) local load transformation matrix; and (3) on option, stress transformation matrix for an axial rod element with unrestrained boundaries.

The stiffness matrix, size 6x6, is in the global coordinate directions. The global coordinate order for each element is (U,V,W) joint 1, then joint 2 where U,V,W are translations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$.

The global load transformation matrix, size 6x6, relates loads at the rod ends in the global coordinate directions to deflections in the global coordinate directions. The row order in this matrix is (P_U, P_V, P_W) joint 1, then joint 2 where P is force.

The local load transformation matrix, size 2x6, relates loads at the rod ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is P_{x1}, P_{x2} where P_x is axial force.

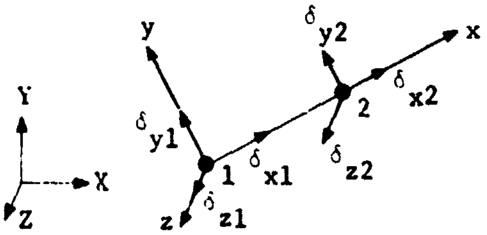
The stress transformation matrix, size 2x6, relates stresses at the rod ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is σ_{x1}, σ_{x2} where σ is normal stress.

These matrices are computed by first calculating a stiffness matrix and stress transformation matrix in the local coordinate system. A direction cosine matrix is then used to transform the stiffness matrix and, on option, the stress transformation matrix from the local coordinate system to the global coordinate directions.

DESCRIPTION OF TECHNIQUE

The calculation of the stiffness matrix, load transformation matrix, and stress transformation in the global coordinate directions is accomplished as follows. First a stiffness matrix is calculated in the local coordinate system and can be given as $[k_L] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ (reference subroutine K1A1).

A sketch of the rod is given for reference as



The strain energy using local coordinates is

$$U = \frac{1}{2}\,[\text{same as column}]^T \left[\begin{array}{ccc|ccc} k_{11} & & & k_{12} & & \\ & 0 & & & 0 & \\ & & 0 & & & 0 \\ \hline k_{21} & & & k_{22} & & \\ & 0 & & & 0 & \\ & & 0 & & & 0 \end{array}\right] \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \delta_{z1} \\ \delta_{x2} \\ \delta_{y2} \\ \delta_{z2} \end{bmatrix}$$

$$= \frac{1}{2}\,[\text{same as column}]^T \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \delta_{x1} \\ \delta_{x2} \end{bmatrix} \qquad [1]$$

The deflections in the local system are related to the deflections in the global coordinate directions by

$$\begin{bmatrix} \delta_{x1} \\ \delta_{x2} \end{bmatrix} = \{h_L\} = [\gamma]\,\{h_G\} \qquad [2]$$

where $[\gamma]$ is a direction cosine matrix (reference subroutine DCOS1A) including Euler angles, size 2x6, and

$\{h_G\}^T = [U_1\ V_1\ W_1\ U_2\ V_2\ W_2]$. U, V, W are translations. Substituting Eq [2] into [1] gives

$$U = \frac{1}{2}\,\{h_G\}^T\,[k_G]\,\{h_G\} \qquad [3]$$

where $[k_G] = [\gamma]^T\,[k_L]\,[\gamma]$ is the stiffness matrix in global coordinate directions.

The loads in the global coordinate directions can be calculated from Eq [3] as

$$\{p_G\} = \frac{\partial U}{\partial\,\{h_G\}} = [k_G]\,\{h_G\} \qquad [4]$$

Thus $[k_G]$ is also a global load transformation matrix giving loads in the global coordinate directions to deflections in the global coordinate directions.

The loads in the local coordinate directions can be calculated from Eq [1] as

$$\{p_L\} = \frac{\partial U}{\partial \{h_L\}} = [k_L]\{h_L\} \tag{5}$$

Substituting Eq [2] gives

$$\{p_L\} = [TL]\{h_G\} \tag{6}$$

where $[TL] = [k_L][\gamma]$ is the local load transformation matrix giving the loads in local coordinate directions to deflections in the global coordinate directions.

A stress transformation matrix relating stresses in the local coordinate directions to deflections in the local coordinate directions is first calculated (reference subroutine K1A1), that is,

$$\{s_L\} = [TS_L]\{h_L\} . \tag{7}$$

On option, the stress transformation matrix relating stresses in the local coordinate directions to deflections in the global coordinate directions is calculated. Substituting Eq [2] into [7] gives

$$\{s_L\} = [TS]\{h_G\}$$

where

$$[TS] = [TS_L][\gamma] .$$

Subroutine STF1B calculates a finite element: (1) stiffness matrix (same as global load transformation matrix); (2) local load transformation matrix; and (3) on option, stress transformation matrix for a combined axial-torsion-bending bar element with unrestrained boundaries.

The stiffness matrix, size 12x12, is in the global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1; then joint 2 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

The global load transformation matrix, size 12x12, relates loads at the bar ends in the global coordinate directions to deflections in the global coordinate directions. The row order in this matrix is (P_U, P_V, P_W, M_P, M_Q, M_R) joint 1; then joint 2 where P is force and M is moment.

The local load transformation matrix, size 12x12, relates loads at the bar ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is P_{x1}, P_{x2}, M_{x1}, M_{x2}, P_{y1}, P_{y2}, M_{z1}, M_{z2}, P_{z1}, P_{z1}, M_{y1}, M_{y2}, where P is force and M is moment.

The stress transformation matrix, size 12x12, relates stresses at the bar ends in the local coordinate system to deflections in the global coordinate directions. The row order in this matrix is

$$\frac{P_{x1}}{A_1},\ \frac{P_{x2}}{A_2},\ \frac{M_{x1} * r_1}{J_1},\ \frac{M_{x2} * r_2}{J_2},$$

$$\frac{P_{y1}}{A_1},\ \frac{P_{y2}}{A_2},\ \frac{M_{z1} * c_{y1}}{I_{z1}},\ \frac{M_{z2} * c_{y2}}{I_{z2}}$$

$$\frac{P_{z1}}{A_1},\ \frac{P_{z2}}{A_2},\ \frac{M_{y1} * c_{z1}}{I_{y1}},\ \frac{M_{y2} * c_{z2}}{I_{y2}}$$

where

P is force,

M is moment,

A is cross-sectional area

r is distance from torsion axis to outer fiber

J is cross-section Saint Venant's torsion constant JG

c is distance from bending neutral plane to outer fiber and

I is area moment of inertia about local axis.

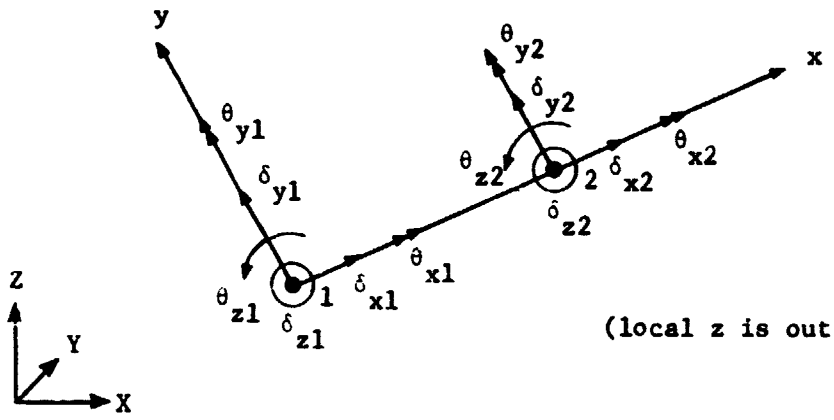
These matrices are computed by first calculating a stiffness matrix and stress transformation matrix in the local coordinate system. A direction cosine matrix is then used to transform the stiffness matrix and, on option, the stress transformation matrix from the local coordinate system to the global coordinate directions.

DESCRIPTION OF TECHNIQUE

The calculation of the stiffness matrix, load transformation matrix, and stress transformation in the global coordinate directions is accomplished as follows. First a stiffness matrix is calculated in the local coordinate system using uncoupled axial, torsion, and bending subroutines listed.

	Subroutine
axial	K1A1
bending	K1B1
torsion	K1C1

A sketch of the bar is given for reference as



(local z is out from paper)

Strain energy using local coordinates is

$$U = \frac{1}{2} \{h_L\} \; [k_L] \; \{h_L\} \qquad [1]$$

where

$$\{h_L\}^T = [\delta_{x1} \; \delta_{x2} \; \vdots \; \theta_{x1} \; \theta_{x2} \; \vdots \; \delta_{y1} \; \delta_{y2} \; \theta_{z1} \; \theta_{z2} \; \vdots \; \delta_{z1} \; \delta_{z2} \; \theta_{y1} \; \theta_{y2}]$$

and

$$[k_L] = \begin{bmatrix}
a_{11} & a_{12} & & & & & & & & & & \\
a_{21} & a_{22} & & & & & & & & & & \\
 & & t_{11} & t_{12} & & & & & & & & \\
 & & t_{21} & t_{22} & & & & & & & & \\
 & & & & b_{11} & b_{12} & -b_{13} & -b_{14} & & & & \\
 & & & & b_{21} & b_{22} & -b_{23} & -b_{24} & & & & \\
 & & & & -b_{31} & -b_{32} & b_{33} & b_{34} & & & & \\
 & & & & -b_{41} & -b_{42} & b_{43} & b_{44} & & & & \\
 & & & & & & & & b_{11} & b_{12} & b_{13} & b_{14} \\
 & & & & & & & & b_{21} & b_{22} & b_{23} & b_{24} \\
 & & & & & & & & b_{31} & b_{32} & b_{33} & b_{34} \\
 & & & & & & & & b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix}$$

a_{ij}, t_{ij}, b_{ij} refer to terms in uncoupled axial, torison, bending stiffness matrices.

Deflections in the local system are related to deflections in the global coordinate directions by

$$\{h_L\} = [\gamma]\,\{h_G\} \qquad [2]$$

where $[\gamma]$ is a direction cosine matrix (reference subroutine DCOS1B) including Euler angles, size 12x12, and

$\{h_G\}^T = [U_1\ V_1\ W_1\ P_1\ Q_1\ R_1\ U_2\ V_2\ W_2\ P_2\ Q_2\ R_2]$. U, V, W are translations and P, Q, R are rotations.

Substituting Eq [2] into [1] gives

$$U = \frac{1}{2}\,\{h_G\}^T\,[k_G]\,\{h_G\} \qquad [3]$$

where $[k_G] = [\gamma]^T\,[k_L]\,[\gamma]$ is the stiffness matrix in global coordinate directions.

Loads in global coordinate directions can be calculated from Eq [3] as

$$\{p_G\} = \frac{\partial U}{\partial \{h_G\}} \quad [k_G] \{h_G\} \qquad [4]$$

Thus, $[k_G]$ is also a global load transformation matrix giving loads in the global coordinate directions to deflections in the global coordinate directions.

Loads in local coordinate directions can be calculated from Eq [1] as

$$\{p_L\} = \frac{\partial U}{\partial \{h_L\}} \quad [k_L] \{h_L\} \qquad [5]$$

Substituting Eq [2] gives

$$\{p_L\} = [TL] \{h_G\} \qquad [6]$$

Where $[TL] = [k_L] [\gamma]$ is the local load transformation matrix giving the loads in local coordinate directions to deflections in the global coordinate directions.

A stress transformation matrix relating stresses in local coordinate directions to deflections in local coordinate directions is first calculated (reference subroutines K1A1, K1B1, K1C), that is,

$$\{s_L\} = [TS_L] \{h_L\} \ . \qquad [7]$$

On option, the stress transformation matrix relating stresses in local coordinate directions to deflections in global coordinate directions is calculated. Substituting Eq [2] into [7] gives

$$\{s_L\} = [TS]\{h_G\}$$

where

$$[TS] = [TS_L] [\gamma].$$

Subroutine STF2 calculates a finite element: (1) stiffness matrix (same as global load transformation matrix); (2) local load transformation matrix; and (3) on option, stress transformation matrix for a combined membrane-bending triangle plate element with unrestrained boundaries.

The stiffness matrix, size 18x18, is in the global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1; then joints 2 and 3 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

The global load transformation matrix, size 18x18, relates loads at the triangle vertices in global coordinate directions to deflections in global coordinate directions. The row order in this matrix is $(P_U, P_V, P_W, M_P, M_Q, M_R)$ joint 1; then joints 2 and 3 where P is force and M is moment.

The local load transformation matrix, size 18x18, relates loads at the triangle vertices in the local coordinate system to deflections in global coordinate directions. The row order in this matrix is (P_x, P_y, M_z) joint 1; then joints 2 and 3, next (P_z, M_x, M_y) joint 1; then joints 2 and 3 where P is force and M is moment.

The stress transformation matrix, size 18x18, relates stresses at the triangle vertices in the local coordinate system to deflections in global coordinate directions. The row order in this matrix is $(\sigma_x, \sigma_y, \tau_{xy})$ at $Z = + 1/2t_{ben}$ for joint 1; then joints 2 and 3. Then the same data at $Z = - 1/2t_{ben}$. σ is normal stress and τ is shear stress.

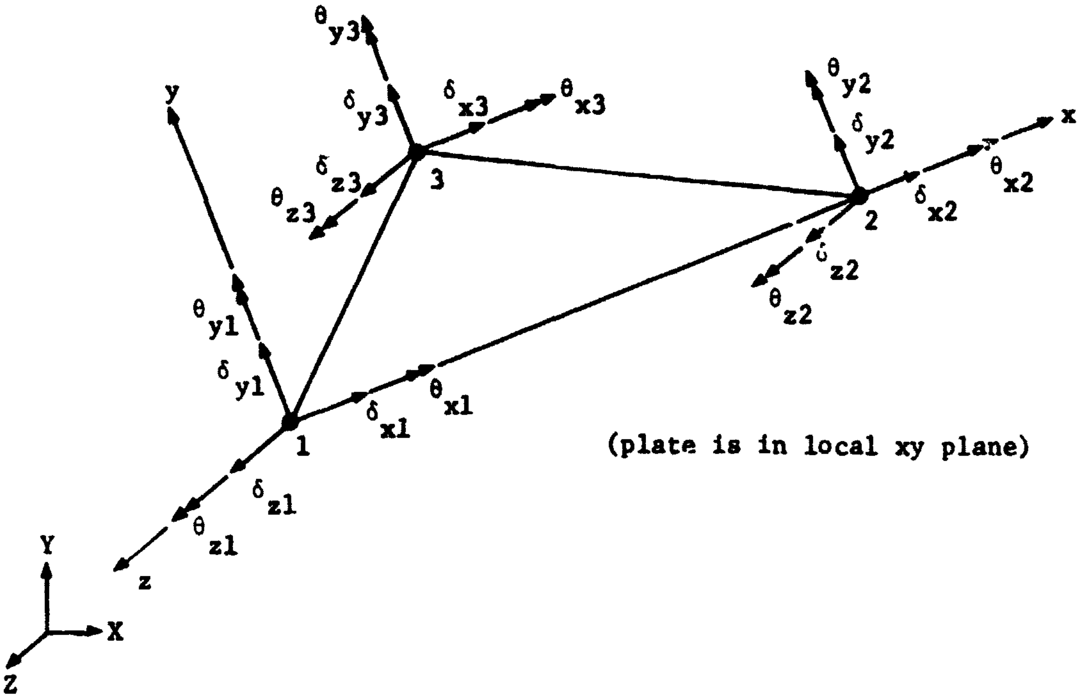
These matrices are computed by first calculating a stiffness matrix and stress transformation matrix in the local coordinate system. A direction cosine matrix is then used to transform the stiffness matrix and, on option, the stress transformation matrix from the local coordinate system to the global coordinate directions.

DESCRIPTION OF TECHNIQUE

The calculation of the stiffness matrix, load transformation matrix, and stress transformation in global coordinate directions is accomplished as follows. First a stiffness matrix is calculated in the local coordinate system using the uncoupled membrane and bending subroutines listed.

	Subroutine
membrane	K2A1
bending	K2B1

A sketch of the triangle plate is given for reference as



The strain energy using local coordinates is

$$U = \frac{1}{2} \{h_L\}^T [k_L] \{h_L\} \qquad [1]$$

where

$$\{h_L\}^T = [\delta_{x1}\ \delta_{y1}\ \theta_{z1}\quad \delta_{x2}\ \delta_{y2}\ \theta_{z2}\quad \delta_{x3}\ \delta_{y3}\ \theta_{z3}$$
$$\delta_{z1}\ \theta_{x1}\ \theta_{y1}\quad \delta_{z2}\ \theta_{x2}\ \theta_{y2}\quad \delta_{z3}\ \theta_{x3}\ \theta_{y}\ ]$$

and

$$[k_L] = \left[\begin{array}{c:c} [k_L]_{mem} & 0 \\ \hdashline 0 & [k_L]_{ben} \end{array}\right]$$

Deflections in the local system are related to the deflections in global coordinate directions by

$$\begin{bmatrix} \delta_{x1} \\ \delta_{x2} \end{bmatrix} = \{h_L\} = [\gamma] \{h_G\} \qquad [2]$$

where $[\gamma]$ is a direction cosine matrix (reference subroutine DCOS2) including Euler angles, size 18x18, and

$$\{h_G\}^T = [U_1 \ V_1 \ W_1 \ P_1 \ Q_1 \ R_1 \ U_2 \ V_2 \ W_2 \ P_2 \ Q_2 \ R_2].$$

U, V, W are translations; P, Q, R are rotations.

Substituting Eq [2] into [1] gives

$$U = \frac{1}{2} \{h_G\}^T [k_G] \{h_G\} \qquad [3]$$

where $[k_G] = [\gamma]^T [k_L] [\gamma]$ is the stiffness matrix in global coordinate directions.

Loads in the global coordinate directions can be calculated from Eq [3] as

$$\{P_G\} = \frac{\partial U}{\partial \{h_G\}} = [k_G] \{h_G\} \qquad [4]$$

Thus, $[k_G]$ is also a global load transformation matrix giving loads in global coordinate directions to deflections in global coordinate directions.

Loads in the local coordinate directions can be calculated from Eq [1] as

$$\{P_L\} = \frac{\partial U}{\partial \{h_L\}} = [k_L] \{h_L\} \qquad [5]$$

Substituting Eq [2] gives

$$\{P_L\} = [TL] \{h_G\} \qquad [6]$$

where $[TL] = [k_L] [\gamma]$ is the local load transformation matrix giving the loads in local coordinate directions to deflections in global coordinate directions.

A stress transformation matrix relating stresses in the local coordinate directions to deflections in the local coordinate directions is first calculated (reference subroutines K2A1, K2B1), that is,

$$\{s_L\} = [TS_L]\ \{h_L\}\ . \qquad [7]$$

On option, the stress transformation matrix relating stresses in local coordinate directions to deflections in global coordinate directions is calculated. Substituting Eq [2] into [7] gives

$$\{s_L\} = [TS]\{h_G\}$$

where

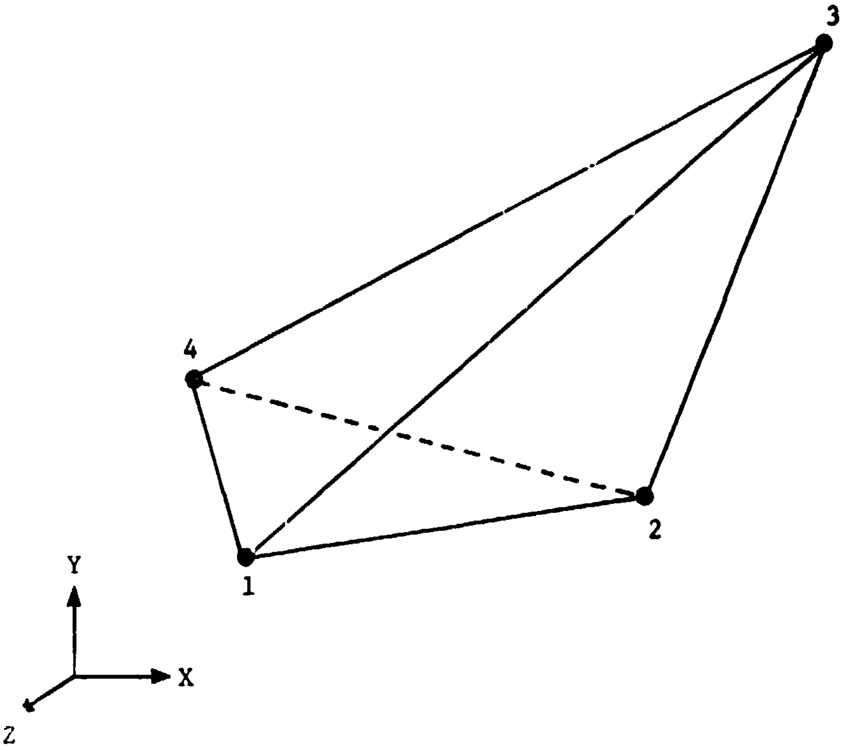
$$[TS] = [TS_L]\ [\gamma].$$

Subroutine STF3 calculates a finite element stiffness matrix (same as global load transformation matrix) for a combined membrane-bending quadrilateral plate element with unrestrained boundaries.

The stiffness matrix, size 24x24, is in global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1; then joints 2, 3 and 4 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

Each global load transformation matrix, size 24x24, relates loads at the quadrilateral vertices in global coordinate directions to deflections in global coordinate directions. The row order in this matrix is (P_U, P_V, P_W, M_P, M_Q, M_R) joint 1; then joints 2, 3 and 4 where P is force and M is moment.

This stiffness matrix is calculated by taking the average overlap of four triangles, shown in the sketch. Subroutine STF2 is used for calculation of the stiffness matrix for the triangular plates.



Subroutine TRNGL calculates (on option) finite element: (1) mass matrices, (2) stiffness matrices (same as global load transformation matrices); (3) local load transformation matrices; (4) stress transformation matrices; and (5) vectors to locate the DOF (degrees of freedom) of these matrices in the global DOF, for combined membrane-bending triangle plate elements. These matrices and vectors are written on disk units and constitute the output from this subroutine. All matrices are in dense programming logic.

Each mass and stiffness matrix, size 18x18, is in global coordinate directions. The global coordinate order for each element is (U, V, W, P, Q, R) joint 1; then joints 2 and 3 where U, V, W are translations and P, Q, R are rotations. If the Euler angles are zero at a joint, then $U=\delta_X$, $V=\delta_Y$, $W=\delta_Z$, $P=\theta_X$, $Q=\theta_Y$, $R=\theta_Z$.

Each global load transformation matrix, size 18x18, relates loads at the triangle vertices in global coordinate directions to deflections in global coordinate directions. The row order in this matrix is (P_U, P_V, P_W, M_P, M_Q, M_R) joint 1; then joints 2 and 3 where P is force and M is moment.

Each local load transformation matrix, size 18x18, relates loads at the triangle vertices in the local coordinate system to deflections in global coordinate directions. The row order in this matrix is (P_x, P_y, M_z) joint 1; then joints 2 and 3 next (P_z, M_x, M_y) joint 1; then joints 2 and 3 where P is force and M is moment.

Each stress transformation matrix, size 18x18, relates stresses at the triangle vertices in the local coordinate system to deflections in global coordinate directions. The row order in this matrix is (σ_x, σ_y, τ_{xy}) at $Z = +1/2t_{ben}$ for joint 1; then joints 2 and 3; then the same data at $Z = -1/2t_{ben}$. σ is normal stress and τ is shear stress.

Each location vector (IVEC) locates the DOF of each finite element in the global DOF. For example, IVEC(6)=834 places element DOF 6 into global DOF 834. IVEC(3)=0 [illegible] element DOF 3 from global DOF. This constrains element DOF 3 to zero motion.

The above matrices are calculated by using joint data and element data. The joint data, obtained from three matrices input to this subroutine, are (1) joint global X, Y, Z locations, (2) joint global DOF numbers, and (3) joint Euler angles.

The element data read in this subroutine is (1) options for mass, stiffness, local load transformations, stress transformations; (2) element material properties; and (3) element joint numbers.

Each mass matrix is calculated by transfer to subroutine MAS2. Each stiffness matrix, loads and stress transformation matrix is calculated by transfer to subroutine STF2.